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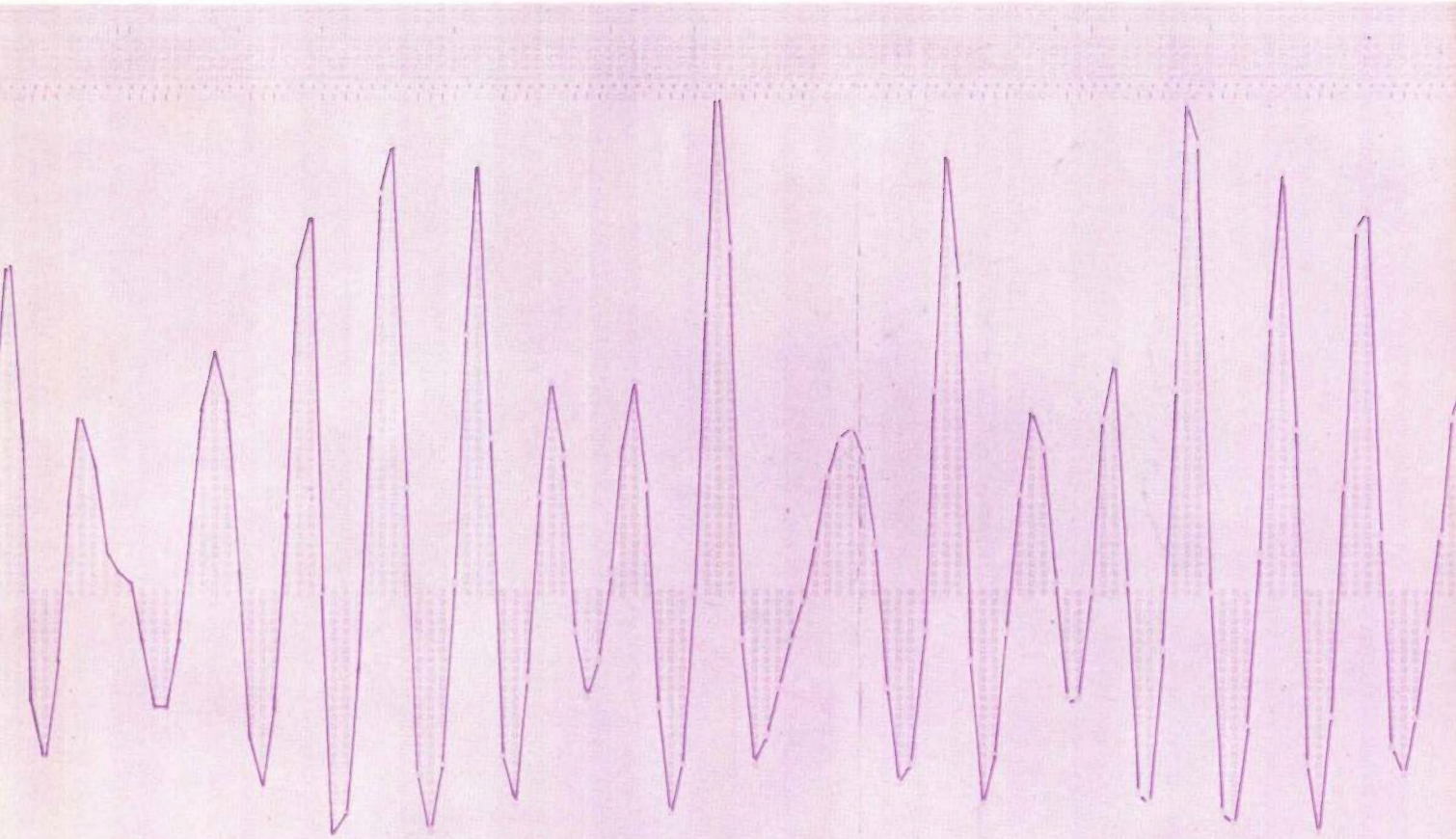

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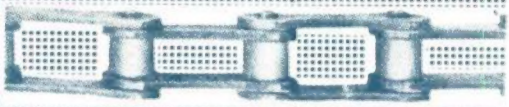

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**CAPA:** — Simulação numérica da velocidade instantânea em escoamento turbulento. Ordenada ≡ V; abcissa ≡ tempo.

A curva corresponde a uma solução do sistema de equações

$$\Sigma_j \frac{\partial u_i}{\partial t} + \partial u_j \frac{\partial u_i}{x_j} = \nu \frac{\partial^2 u_i}{\partial x_j^2} + f(x_j) \; (i = 1, 2, 3)$$

obtido por J. D. Domingos.

Representação gráfica da velocidade num ponto feito através da impressora do sistema IBM 360/44 do Centro da Universidade Técnica de Lisboa usando uma subrotina desenvolvida por S. D. Antunes (trabalho integrado num projecto de Investigação do I. A. C. em curso no Núcleo de Estudos de Engenharia Mecânica).

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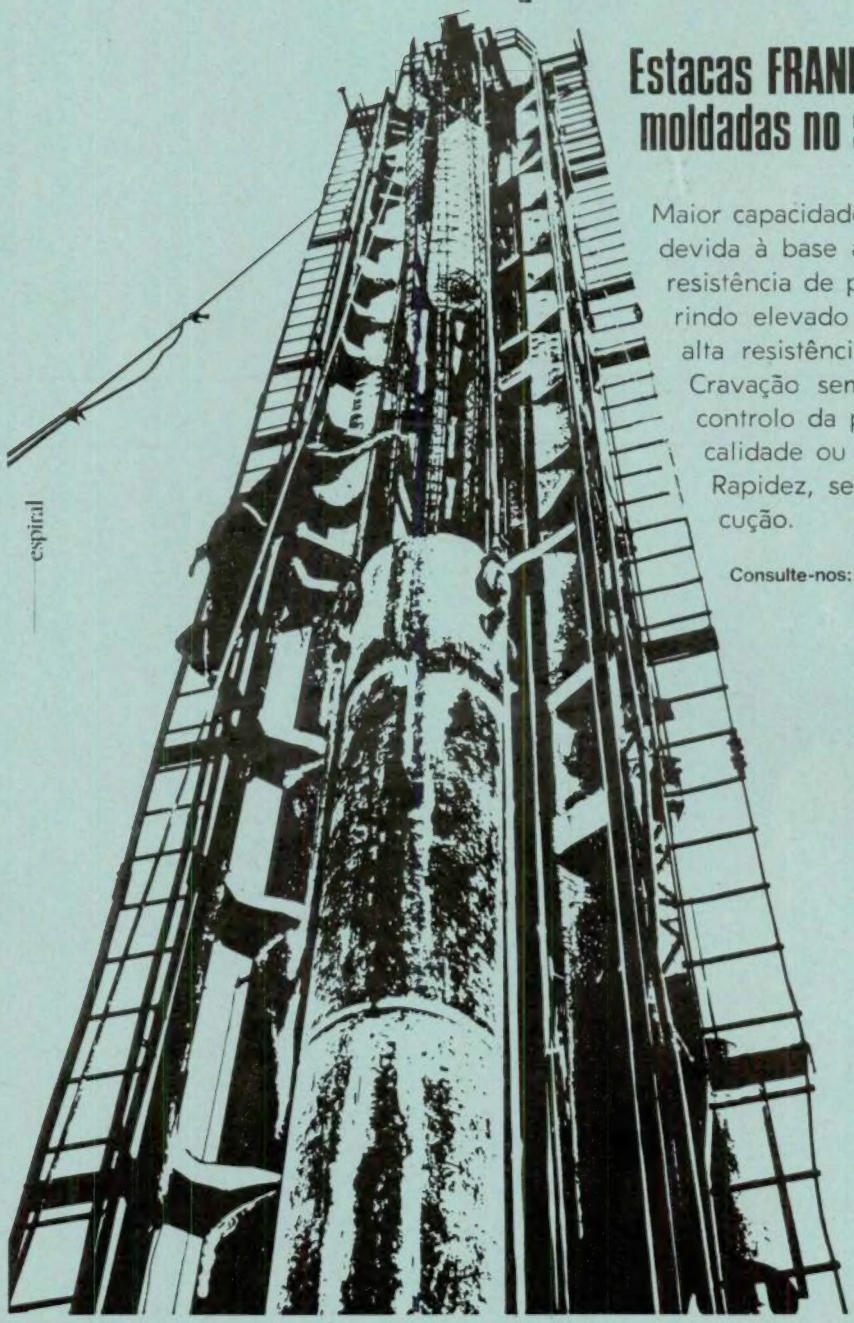

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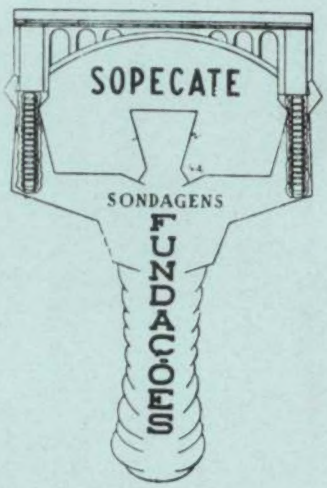


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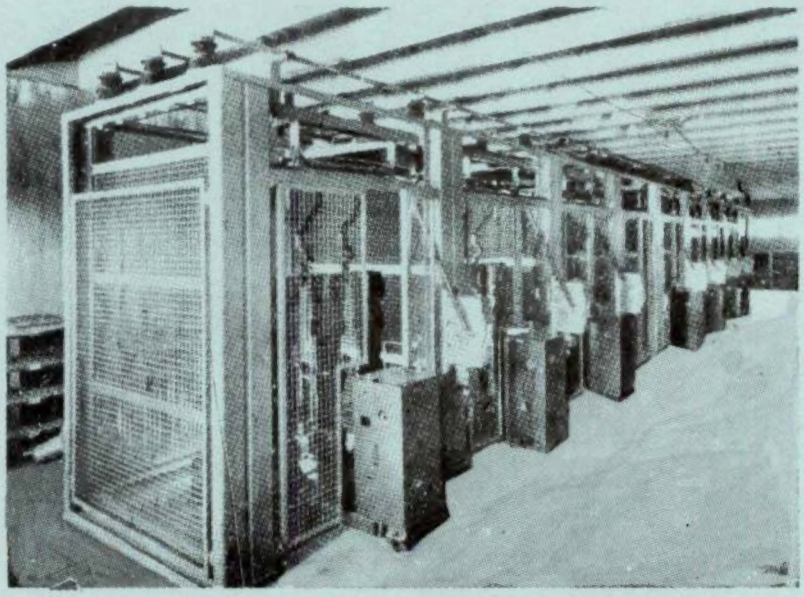

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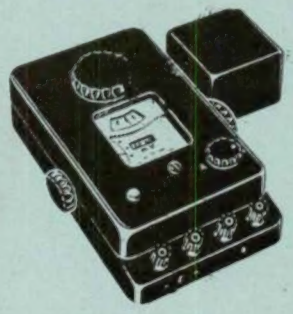

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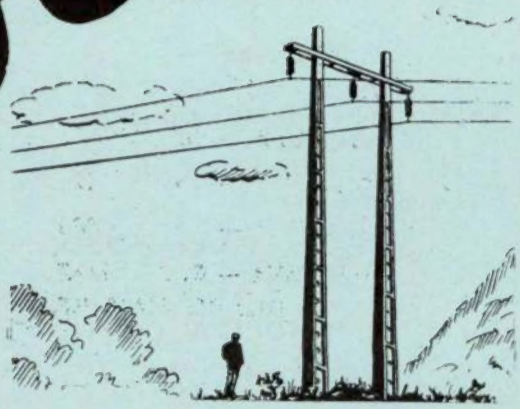

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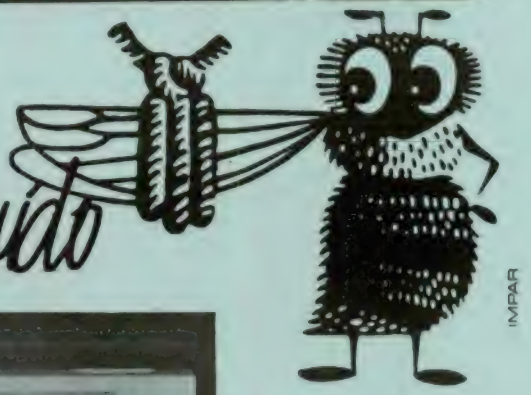

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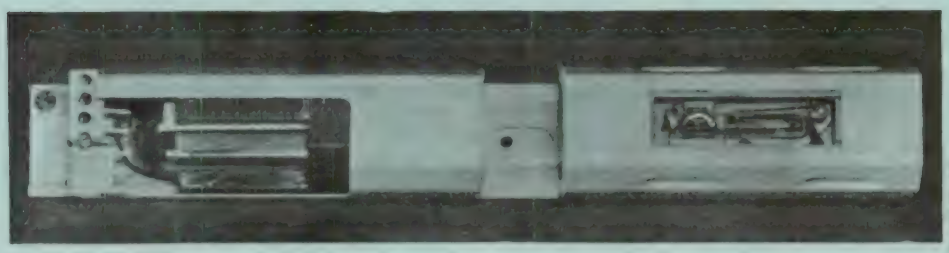

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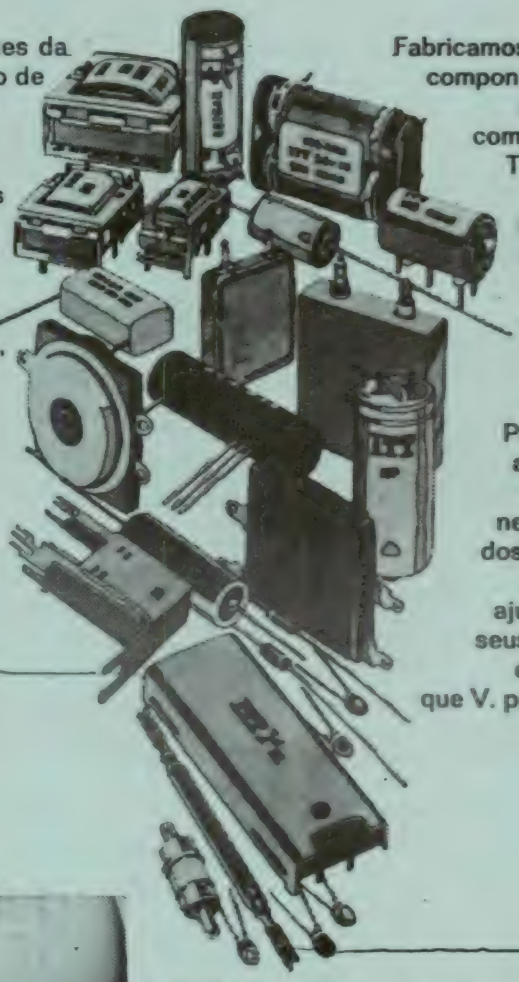

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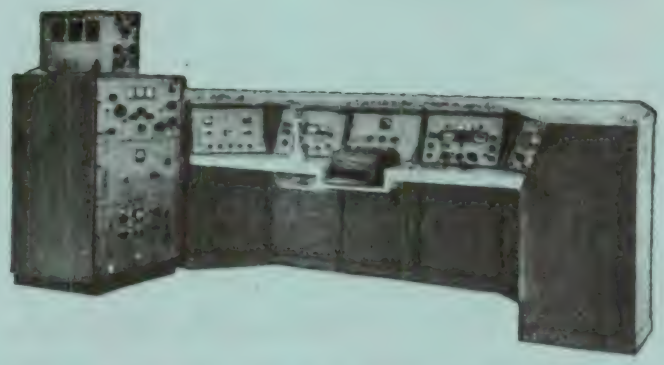

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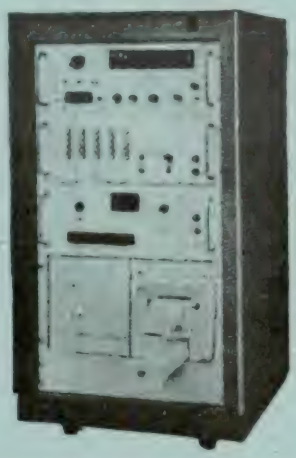

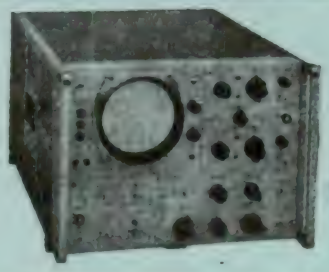

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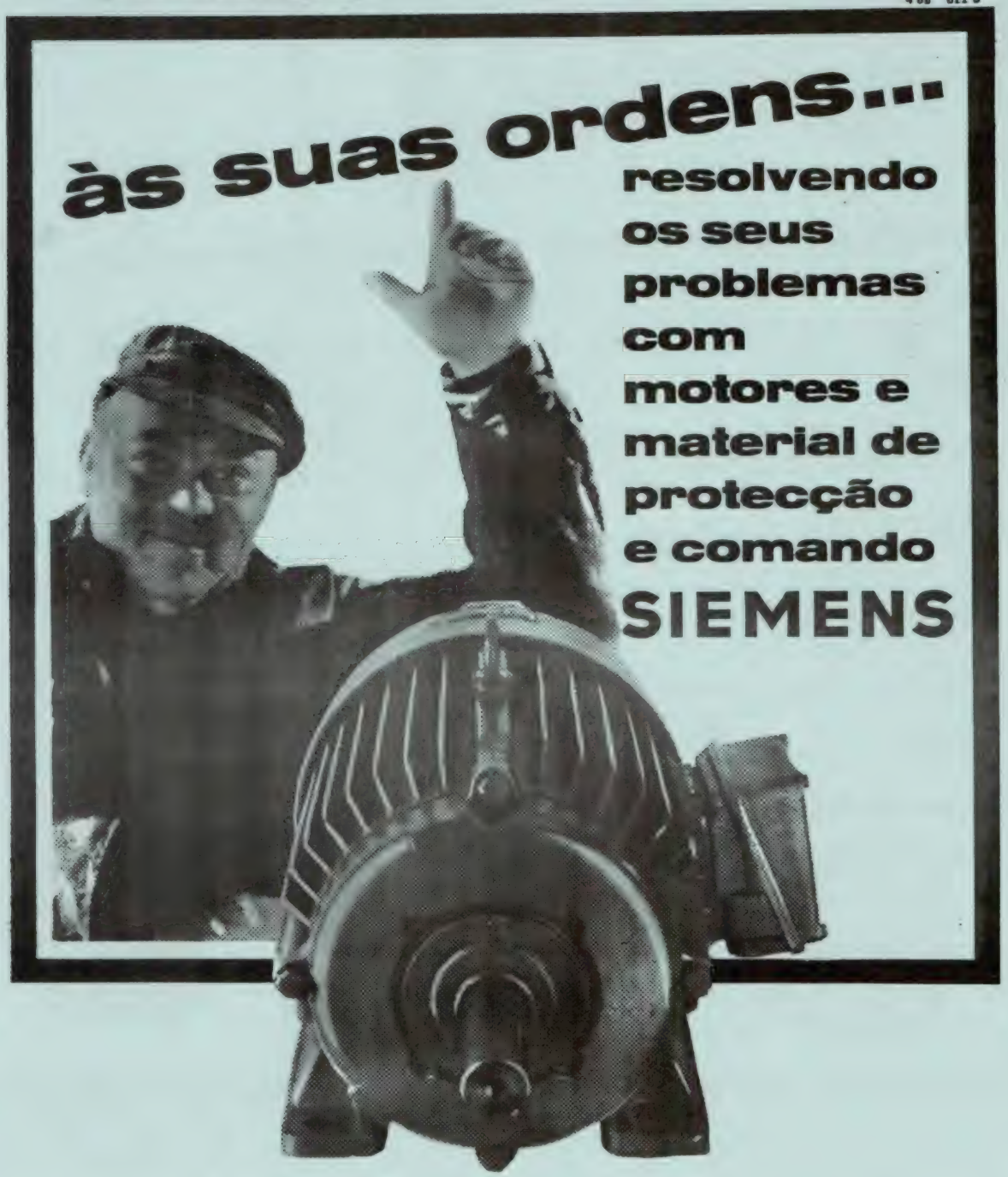


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## INTRODUCTION

0. As an outcome of research work in turbulent flows started at Instituto Superior Técnico, Department of Mechanical Engineering, Division of Applied Thermodynamics (sponsored by Instituto de Alta Cultura and Fomento da Acção Educativa do M. E. N.) with the close cooperation of Imperial College, (University of London, Department of Mechanical Engineering, Heat Transfer Section) and especially of Dr. J. H. Whitelaw, an extension course on «Heat and Mass Transfer in Turbulent Bondary-layers» was given at Instituto Superior Técnico in July 1969, by Dr. J. H. Whitelaw (I. C.) and J. J. D. Domingos.

This course, centered on a presentation of the Spalding-Patankar method of computation of turbulent boundary layers together with computational results of Engineering importance, was a modified form of a post-experience course given at Imperial College in December 1968 and December 1969. The main contents of this course are now widely known through ref. (2)*.

Due to the success of the course of July 1969 at I. S. T., a second course was evolved for 1970 specifically to satisfy interests of the Lisbon audience, and to require a greater contribution from the local research group. Because much of the content of this course is not available in published form and because it was found desirable for future courses to have it available, the more inaccessible material is to be published in «Técnica» having in mind the wider audience which can benefit from the material and at the same time, became acquainted with work presently in progress at I. S. T.

0.1. The papers which follow (Parts I, II, III, IV) closely resemble the lecture notes specially prepared for the course. The presentation could undoubtedly be reworked to improve clarity and comprehensiveness but this would imply delaying publication for an even longer time than has already elapsed. Neverthless, we expect the material to be useful to potential readers who may consider it as introduction to a increasingly important subject. If, as we hope, new courses follow the two already given, we confidently expect this publication to be a forerunner for more ambitious ones.

0.2. As originally given, the course included lectures and experimental demonstrations which included the preparation of hot wire probes and manipulation of the equipment by participants.

The lectures where given by J. J. D. Domingos (I. S. T.), J. H. Whitelaw (I. C.), F. Durst (I. C.) and S. D. Antunes (I. S. T.) and the experimental sections were conducted by F. Durst, S. D. Antunes and M. N. Mina.

1. Once the «how» is explained, if the patience of the reader is not already exausted, the «why» of the contents is possibly worth further comment.

The «why» comes from what we believe must be known by newcomers to the field, and what we think should be remembered by experienced readers.

For the mathematician, turbulent flow in incompressible fluids only require the solution of

* References collected at the end of part II.

a set of four quasi-linear parabolic equations known for more than one hundred years but which no-one has yet been able to solve without drastic simplifications. And from the time of Poisson, Navier and Stokes the best mathematicians in each generation have abandoned the problem as untractable and apparently unsolvable.

For the physicist, turbulent flow is the only great unsolved problem of classical physics . . .

The engineer, however, is obliged to consider fluid flow in many aspects of daily routine. He designes aeroplanes, tall structures taking account of wind loads, turbomachinery, dams, space vehicles, artificial heart-lung machines, heat--exchangers, combustion equipment, etc. with the constant challenge to produce better and cheaper designs. He can not abandon these problems. In the good old days, crude experiments and empiricism were enough. Later, dimensional analysis helped. Today is still the main tool . . . but no longer sufices.

Yet, the potencial means of solving turbulent flow problems is still the solution of a set of four partial differential equations, or five, or six, or any number, if we add compressibility, heat transfer, chemical reaction . . . for which a complete proof of physical validity is still to be produced.

If one remember this, and also that such famous names as Landau, Onsager, Sommerfeld, G. I. Taylor, Kolmogorov and Kampé Forrié (some of them Nobel prize winners) abandoned the problem for more fruitfull areas of fundamental research, one must refrain from a frontal attack on some crude simplifications currently in use.

There is only aestectic value, if any, in proposing solutions in terms of constants, or integrals etc, whose translation to numerical answers is beyond human reach; they do not give any fundamental insight into the physical mechanisms involved, besides the personal believe of the proponent in his own physical intuition.

Summing up, we would like to escape in the 20th century what Sir Cyril Hinshelwood said about 19th century: «Fluid dynamicists were divided into hydraulic enginners who observed what could not be explained, and mathematicians who explained things that could not be observed».

*J. J. D. Domingos*

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# I — THE FUNDAMENTAL EQUATIONS OF TURBULENT INCOMPRESSIBLE FLOWS

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RESUMO

*Deduzem-se as equações fundamentais para o estudo dos escoamentos turbulentos incompressiveis e discutem se as hipóteses fundamentais e suas limitações.*

*Apresenta-se um resumo sucinto da posição actual da teoria matemática das equações de Navier-Stokes e do seu tratamento numérico.*

*Discute-se em pormenor a equação de energia cinética da turbulência e a sua correcta dedução, tendo em conta o papel central que a mesma desempenha na formulação de modelos de turbulência.*

*Todo o trabalho é orientado sob o ponto de vista da engenharia, no sentido do cálculo do escoamento médio.*

SYNOPSIS

*The fundamental equations for the study of turbulent incompressible flows are derived and its underlying assumptions and limitations discussed. Present knowledge regarding mathematical and numerical aspects of the Navier-Stokes equations are briefly presented. Special attention is given to a correct derivation and discussion of the turbulent kinetic energy equation having in mind its central position in the development of turbulence models for prediction purposes.*

*All the paper is oriented towards engineering prediction procedures of mean flow behaviour.*

## 0. BASIC ASSUMPTIONS AND RELATIONS

Except when explicitly stated, we assume:

- homogeneous, incompressible and Newtonian fluids
- isothermal flow
- cartesian frame, fixed to the Laboratory.

The substantive derivative (or derivative following the motion) is expressed by $\frac{d}{dt}$

1) $$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + V_j \frac{\partial}{\partial x_j}$$

where, as everywhere, a repeated index means summation.

The Reynolds transport theorem is expressed by:

2) $$\frac{d\phi}{dt} = \frac{d}{dt}(\rho\phi) + \rho\phi \frac{\partial V_j}{\partial x_j}$$

and mass conservation by:

3) $$\frac{dV_j}{dx_j} \equiv \frac{\partial V_j}{\partial x_j} = 0 \quad \text{(because of incompressibility)}$$

### 0.1. General form of a balance equation:

4) $$\frac{d\phi}{dt} = -\frac{\partial}{\partial x_j}(J_\phi)_j - \rho Q_\phi$$

Usually:

$Q_\phi = G - \varepsilon$

with $G > 0$ — production of $\phi$ (positive source)

$\varepsilon > 0$ — dissipation of $\phi$ (negative source)

if

$Q_\phi = G - \varepsilon = 0$, $\phi$ is a conserved property

For a volume element fixed in space (Euler description), we have, from 4) and 1)

6) $$\frac{\partial\phi}{\partial t} = -\frac{\partial}{\partial x_j}(V_j\phi + J_{\phi_j}) - \rho Q_\phi$$

where use has been made of the identity

7) $$V_j \frac{\partial\phi}{\partial x_j} = \frac{\partial}{\partial x_j}(\phi V_j) - \phi \frac{\partial V_j}{\partial x_j}$$

or 8) $$V_j \frac{\partial\phi}{\partial x_j} = \frac{\partial}{\partial x_j}(\phi V_j)$$

because of 3).

The term $\phi V_j$ is the convective (or advective) flux of $\phi$, and is purely due to the fluid motion,

while $J_{\phi_j}$ is a flux of $\phi$ independent of that mass motion.

From 4) and 1)

$$9)\quad \frac{\partial \phi}{\partial t} + V_j \frac{\partial \phi}{\partial x_j} = -\frac{\partial}{\partial x_j}(J_\phi)_j - \rho(G-\varepsilon)$$

## 1.0. The Navier-Stokes equations

For a volume element

$$10)\quad \rho \frac{dV_i}{dt} = -\frac{\partial \tau_{ij}}{\partial x_j} + \rho F_i$$

The terms which appear have the following physical interpretation:

$\rho \frac{dV_i}{dt}$ = mass times aceleration in the i direction

$\tau_{ij}$ — stress tensor: stresses applied for unit surface of the volume element (tensorial density)

$-\frac{\partial}{\partial x_j}\tau^{ij}$ — resultant force on direction i of the stresses on the surface

$\rho F_i$ = i component of the applied forces for unit mass

Equation 10) can also be expressed as a momentum balance.

From 2)

$$\rho \frac{dV_i}{dt} = \frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}(\rho V_i V_j)$$

from 10) with 8) and putting $\phi \equiv V_i$

$$11)\quad \frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_j}[\rho V_i V_j + \tau_{ij}] + \rho F_i$$

which expresses the local (Euler) balance of momentum.

Equation (10) or 11) are indiferently used as starting equations.

1.1. For an incompressible Newtonian fluid the stress tensor has the form:

$$\tau_{ij} = P^* \delta_{ij} + \mu\left(\frac{\partial V^i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$$

where $P^*$ is the hydrostatic pressure.

From 11) and 12)

$$13)\quad \frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_j}\left[P^* \delta_{ij} + \rho V_i V_j + \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)\right] + \rho F_i$$

where the terms have the following interpretation:

$\frac{\partial}{\partial t}(\rho V_i)$ — rate of accumulation of momentum in the i direction in the volume element

$P^* \delta_{ij} + \rho V_i V_j$ = reversible flux of momentum

$\mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$ = momentum flux due to viscous forces

For the incompressible fluid with constant viscosity

$$\frac{\partial}{\partial x_j}\mu \frac{\partial V_i}{\partial x_i} = \mu \frac{\partial}{dx_j}\frac{\partial V_j}{\partial x_i} = \mu \frac{\partial}{\partial x_i}\frac{\partial V_j}{\partial x_j} = 0$$

because $\frac{\partial V_j}{\partial x_j} = 0$

So, we can write equation 13) as

$$14)\quad \frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_j}\left(P^* \delta_{ij} + \frac{\partial V_i}{\partial x_j}\right) + \rho F_i$$

or

$$15)\quad \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P^*}{\partial x_i} + \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j} + \rho F_i$$

Equation 14) or 15) with

$$16)\quad \frac{\partial V_j}{\partial x_j} = 0$$

are the basic equations for the motion of the Newtonian incompressible fluid, or, in short, the N.S. equations.

## 2. THE KINETIC ENERGY EQUATION

Some of the terms which appear in the N. S. equations are better understood when the kinetic energy balance is considered

For simplicity we assume $F_i = 0$

The kinetic energy per unit volume is

$$E = \frac{1}{2} \rho V^2 = \frac{1}{2} \rho V_i V_i$$

and its change in time is

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V_i V_i \right) = \rho V_i \frac{\partial V_i}{\partial t}$$

Taking $\frac{\partial V_i}{\partial t}$ from 15) and writing

$$17) \qquad \tau_{ij} = \mu \frac{\partial V_i}{\partial x_j} \; (*)$$

we get

$$18) \qquad \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V_i V_i \right) = -\rho V_i V_j \frac{\partial V_i}{\partial x_j} - V_i \frac{\partial P^*}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

which is the balance equation for the kinetic energy.

Taking into account the identities:

$$V_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} (V_i \tau_{ij}) - \tau_{ij} \frac{\partial V_i}{\partial x_j}$$

$$V_i V_j \frac{\partial V_i}{\partial x_j} = V_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} V_i V_i \right)$$

$$= \frac{\partial}{\partial x_j} \left( V_j \frac{1}{2} V_i V_i \right) - \frac{1}{2} V_i V_i \frac{\partial V_j}{\partial x_j}$$

and, for the incompressible fluid, $\frac{\partial V_j}{\partial x_j} = 0$

$$V_i V_j \frac{\partial V_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left( V_j \frac{V^2}{2} \right)$$

$$V_i \frac{\partial P_i^*}{\partial x_i} = \frac{\partial}{\partial x_i} (P V_j)$$

The kinetic energy balance equation takes the form:

$$19) \quad \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V_i V_i \right) = - \frac{\partial}{\partial x_j} \left[ \rho V_j \left( \frac{1}{2} V_i V_i + \frac{P^*}{\rho} \right) V_i \tau_{ij} \right] - \tau_{ij} \cdot \frac{\partial V_i}{\partial x_j}$$

where the terms have the interpretation:

$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho V_i V_i \right)$ = change of local kinetic energy per unit volume and time.

$\rho V_j \left( \frac{1}{2} V_i V_i + \frac{P^*}{\rho} \right)$ = energy (kinetic plus pressure) flux density in direction j due to the mass flux in the same direction. Is a convection (or advection) of energy.

$V_i \tau_{ij}$ = energy flux density in direction j due to the viscous forces.

The first term on the r. h. s. is the divergence of the flux of the above quantities, and as such represents the difference among outflowing and inflowing quantities. The second term on the r. h. s. is a source term (negative). So

$\tau_{ij} \frac{\partial V_i}{\partial x_j}$ — is the dissipation of kinetic energy for unit volume and time.

As we see, the viscous stresses give rise to two effects: one is an energy flux ($V_i \tau_{ij}$) the other a dissipation.

The energy flux due to viscous stresses results from the fact that the slowers (faster) portions of the fluid in contact with the fasters (slowers) are

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(*) The general relation is

$$\tau_{ij} = \mu \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right), \text{ as is wellknown.}$$

However on the basic equation $\gamma_{ij}$ only enters through divergence $\left( \frac{\partial}{\partial x_j} T_{ij} \right)$ which, for the incompressible fluid with constant viscosity reduce to $\mu \frac{\partial V_i}{\partial x_j}$ as already shown.

This allows one to proceed as if $\gamma_{ij} = \mu \frac{\partial V_i}{\partial x_j}$ with the advantage of avoiding terms that would havet be simplified afterwards.

accelerated (desaccelerated) due to the existence of viscosity. The result is a transfer of momentum which involves, obviously, the simultaneous transfer of kinetic energy.

The term $\tau_{ij} \frac{\partial V_i}{\partial x_j}$ as been called a dissipation due to its non-divergence form in 19). However, to be a dissipation it must be intrinsecally positive. Its positiveness results clearly from:

$$\tau_{ij} \frac{\partial V_i}{\partial x_j} = \mu \frac{\partial V_i}{\partial x_j} = \mu \left(\frac{\partial V_i}{\partial x_j}\right)^2$$

Besides, if we integrate 19) on the whole fluid and if we assume it to be bounded and $V_i = 0$ on the bounding surface, we get, using the divergence theorem:

$$\frac{\partial}{\partial t} \int_v E \, dv = - \mu \int_v \left(\frac{\partial V_i}{\partial x_j}\right)^2 dv$$

which shows that $\mu \left(\frac{\partial V_i}{\partial x_j}\right)^2$ is clearly a dissipation. By the second law of thermodynamics $\mu > 0$.

For future reference we now write 19) taking into account

$$\frac{1}{\rho} \tau_{ij} = \nu \frac{\partial V_i}{\partial x_j}$$

20) $$\frac{\partial}{\partial t}\left(\frac{1}{2} \rho V_i V_i\right) = - \frac{\partial}{\partial x_j} \rho V_j \left(\frac{1}{2} V_i V_i + \frac{P^*}{\rho}\right) - \mu V_i \frac{\partial V_i}{\partial x_j} - \mu \left(\frac{\partial V_i}{\partial x_j}\right)^2$$

or, from 17) and 18)

$$\frac{1}{2} \frac{\partial}{\partial t} V^2 = - \frac{1}{2} V_j \frac{\partial V^2}{\partial x_j} - \frac{V_j}{\rho} \frac{\partial P^*}{\partial x_j} + \frac{\nu}{2} \frac{\partial^2}{\partial x_j^2} V^2 - \nu \left(\frac{\partial V_j^2}{\partial x_j}\right)$$

### 2.1. Kinetic energy balance in a laminar boundary-layer

To clarify the behaviour of the several terms in the kinetic energy equation we will consider now as a typical flow a laminar boundary layer.

Using the typical symbols of the boundary-layer literature we call

$$x_1 = x \qquad V_1 = U$$
$$x_2 = y \quad \text{and} \quad V_2 = V$$
$$x_3 = z \qquad V_3 = W$$

Boundary-layer flows are defined as two dimensional flows where a dominant direction exist such that variations of any flow property are most significant in the cross direction. For plane flows, such «definition» allows (by an order of magnitude argument) the N.S. equations to simplify in plane steady flows to:

$$\frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = - \frac{1}{\rho} \frac{\partial P^*}{\partial x} + \frac{\partial^2 U}{\partial y^2}$$

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 0$$

the flow beeing on the x direction and a wall exists for

$$x > 0 \qquad y = 0$$

if $U = U_\infty = \text{const}$ for $y = \infty$, $x > 0$

$$\frac{dP}{dx} = 0$$

and the equation for the kinetic energy reduces accordingly to

$$\frac{1}{2} U \frac{\partial U^2}{\partial x} + \frac{1}{2} V \frac{\partial U_1^2}{\partial y} - \frac{\nu}{2} \frac{\partial^2 U^2}{\partial y^2} + \nu \left(\frac{\partial U}{\partial y}\right)^2 = 0$$

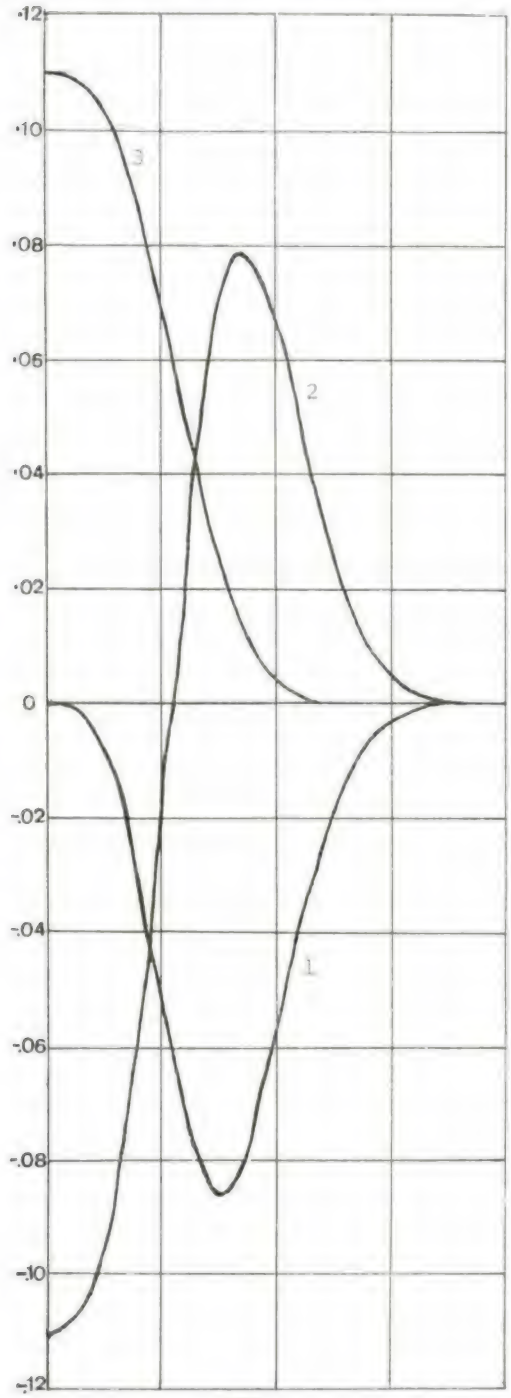
Which we can make nondimensional multiplying by $\frac{x}{U_\infty^3}$

For an initial profile $U = U_\infty$ for $x = 0$ we get the representation of fig. 1

The existence of the wall and of a non sliping condition implies $U = 0$ at $y = 0$ from which results a velocity gradient and a viscous loss, across the flow. However, the non-dimensional viscous dissipation is

$$\frac{x}{U^3} \frac{\nu}{2} \left(\frac{\partial U}{\partial y}\right)^2$$

KINETIC ENERGY BALANCE IN A LAMINAR BOUNDARY LAYER




1) $\frac{x}{U_{\infty 3}} \frac{1}{2} \left( U \frac{\partial U^2}{\partial x} + V \frac{\partial U^2}{\partial y} \right)$

2) $- \frac{x}{U_{\infty 3}} \left( \frac{\nu}{2} \frac{\partial^2 U^2}{dy^2} \right)$

3) $\frac{x}{U_{\infty 3}} \left( \nu \left( \frac{\partial U}{\partial y} \right)^2 \right)$

FIG. 1

and the kinetic energy loss at each point is

$$\frac{x}{U_\infty^3} U \frac{\partial U}{\partial x} + V \frac{\partial U^2}{\partial y}$$

which do not balance everywhere the dissipation. Summing the difference across the layer :

$$\int_0^\infty \frac{x}{U_\infty^3} \frac{\nu}{2} \frac{\partial^2 U}{\partial y^2} dy = 0,$$ so, a global balance exists

between the total loss of kinetic energy and the dissipation, the difusion transfer (curell 2) just taking into account the local unbalance, transfering the kinetic energy from a point to the other.

As fig. 1 shows, the dissipation is maximum at the wall, where the kinetic energy is smallest. The dissipated energy came from the outer portions of the boundary layer where the loss of kinetic energy was in excess of the dissipation.

## 3. SOME MATHEMATICAL RESULTS CONCERNING THE NAVIER-STOKES EQUATION

3.1. The fundamental equations 15) and 16) are a coupled set of four equations with four unknowns : $V_1$, $V_2$, $V_3$, $P^*$.

Mathematically, we have a sistem of quasi--linear partial differential equations of the parabolic type.

These fundamental equations are by themselves far from beeing accepted as truly general and fundamental though many people seems to paraphase Langrange (changing Euler to N. S.) when he wrote in 1788 :

«One owes to Euler the first general formulas for fluid motion ... presented in the simple and luminous notation of partial differences ... By this discovery, all fluid mechanics was reduced to a single point of analysis, and if the equations involved were integrable, one could determine completely, in all cases, the motion of a fluid moved by any forces ... »

Indeed, before general solutions were found the Euler equations shown to be at most an approximation. Our hope have, after, been transfered to the N. S. equations. We have not yet found any general (which mean three dimensional, time dependent) solution, but we know already that they can not be general. They are better than the Euler equation, but we don't

know yet to what extend. However, exact analytical results are of outstanding value even when they concern only incompressible fluids which are a mathematical abstraction without thermodynamic validity.

For incompressible fluids existence and unicity of solution has been prooved for two-dimensional flows. It has also been proved that a steady state solution exists which is the assimptotic limit of a time dependent solution when time goes to $\infty$.

For three-dimensional flows no such proof is known. Under certain restrictive assumptions, unicity of solution exist, and if the flow is completely bounded by a solid wall with non slip conditions, using methods of functional analysis it has been shown that the velocity tends to zero with time: This is a trivial conclusion from a physical point of view as the equation for the kinetic energy shows.

The rigorous proof that N.S. equations admit turbulent solutions has not yet emerged, neither the conclusion that they do not admit, thougth some results point to the first possibility. At least, a very interesting conclusion exists already: two dimensional flows do not admit turbulent solutions which is in complete agreement with the physical experience.

3.2. The almost untractable mathematical problem, with the tools which have been used, does not prevent, however, some interesting conclusions if some additional assumptions are made. These are the object of the following chapters. Before, a comment must be done regarding numerical methods: even with present day computers the solution of the full set of time-dependent equations is impossible — the restrictions put on the stability and accuracy of presently known numerical methods grow so sharply with the Reynolds number that the memory-speed requeriments of the computer go beyond the foreseen future. A crude, lower bound estimate shows these better than words;

Let $\Delta U$, $\Delta x$, $\upsilon$, be the caracteristic velocity variation between two nodes, $\Delta x$ his distance, $\upsilon$ the kinematic viscosity.

Numerical stability requires the cell Reynolds number:

$$\frac{\Delta U \, \Delta x}{\upsilon} < \simeq 1$$

If a flow is divided in 20 nodes in each direction, the number of nodes will be $20 \times 20 \times \times 20 = 8.10^3$. But this number is at most acceptable for completely laminar flow on the Stokes or Oseen region. To go to an acceptable precision an order of magnitude at least is needed, i, e. $200 \times 200 \times 200 = 8.10^6$ nodes. This is on the order of $10^3$ greater than possible with present day computers.

This again shows that one must turn to cruder methods if at least an engineering solution is though which is not the crudest.

The way is to regularise the solutions with physically based guesses.

### 3.3. Turbulent Flow

As is known since the basic experiments of O. Reynolds, above a certain value of the Reynolds number the flow becames unstable. The exact upper bound of this critical Reynolds number is unknown because it depends on the experimental care, surface conditions, inicial profile (in a tube, for exemple, values between 2100 and 70.000 have been found). The lower bound is much better known (i. e, for a tube for Re $<$ 2000 no turbulent flow exists).

Again the question arises as to the validity of the N. S. equations. The stability theory of these equations shows the existence of an instability of the flow in the sense that there are conditions in which a small disturbance is amplified, not damped. The most success full approach to this stability problem has been the «linear stability theory» where the N. S. equations are linearized and on them the evolution in time of a small perturbation considered. However, the growth of the perturbation can not be followed in this theory by the basic reason that when it grows the problem ceasses to be a linear one.

The non-linear theory, which would deal with the problem with not to drastic simplifications, is not yet known to have solved a simple real situation.

Another approach which is slowly emerging is the so-called bifurcation theory.

Bifurcation theory on its present form regarding the N. S. equations is much more conceptual than mathematical, but is worth mentioning because it clarifies, at least, some common concepts.

# MAGUE

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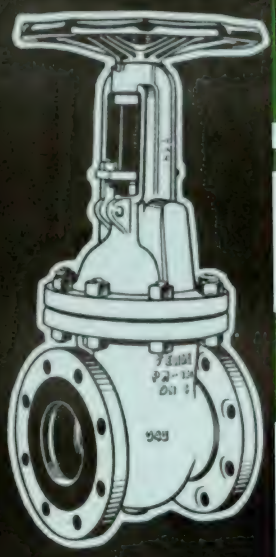

Segundo normas DIN e ASA,
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Execução em ferro fundido, aço
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e ligas especiais.
Diâmetros nominais de 10 a 600mm
Pressões nominais de 4 a 40 kg./cm².

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This theory assumes that for a certain $R_e < R_C$ the first unstable mode of the linear theory will not grow exponentialy (as the linear theory would imply) with time but would tend assymptotically to a value proportional to $(R - R_C)^{1/2}$.

So, at $R > R_C$, two solutions of the N. S. are possible: one, the laminar solution, the other a periodic one of small amplitude, proportional to $(R - R_C)^{1/2}$.

This situation will continue until a second critical Reynolds number, $R_{C_2}$ is reached, where from the laminar solution a second unstable mode grows. The process is repeated for each of the possible solutions.

In this concept, there are not one but a growing number of critical Reynolds numbers, with $R_e$. In this way, for a Reynolds number well above the first critical one the number of allowable solutions of the N. S. equations is enourmous for the given set of boundary conditions, with the flow possibly switching from one to other continuously. From the experimental (macroscopic) point of view, the flow appears as completely random; this is the region of fully develloped turbulence.

On the sequence of the exposition we will always assume fully developed turbulence and will describe the flow by its mean properties. Besides, we will assume the N. S. equations as still valid, and accordingly will introduce in them an averaging process.

The averaging process must be considered more of an heuristic than a rigorous mathematical operation.

we define:*

$$U_i = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} V_i \, dt = \overline{V}_i$$

$$u_i = U_i - V_i \quad \text{and} \quad x_i = 0$$

$$P = \frac{1}{T} \int_{t+\Delta t/2}^{t+\Delta t/2} P^* \, dt = \overline{P}^*$$

$$p = P^* - P \qquad \text{and} \quad p = 0$$

If we do the substitution

$$V_i = U_i + u_i$$

$$P^* = P + p$$

in the N. S. equations (and this is possible on mathematical grounds whaever the definition of $U_i$ or $u_i \ldots$, because we are only doubling the number of unknowns):

$$22) \quad \frac{\partial (U_i + U_i)}{\partial t} + (U_j + u_j) \frac{\partial (U_i + u_i)}{d x_j} =$$

$$- \frac{1}{\rho} \frac{\partial}{\partial x_j} (P + p) + \nu \frac{\partial^2}{\partial x_j \partial x_j} (U_i + u_i)$$

$$\frac{\partial}{\partial x_j} (U_j + u_j) = 0$$

integrating now between t and t + T and assuming the averaging concept as acceptable and taking account that $\frac{\partial}{\partial x}$ and $\int_{t-\Delta t/2}^{t+\Delta t/2}$ are comutative operations (assuming continuity) we get,

$$23) \quad \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \overline{u_j \frac{\partial u_i}{\partial x_j}} =$$

$$- \frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j}$$

$$\frac{\partial U_j}{\partial x_j} = 0$$

where the overbar, denotes time average, alway.

Taking into account that

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial}{\partial x_j} \overline{(u_j x_i)}$$

* we could have defined an «ensemble average» coined after the statistical mechanics concept. This would be pretensious and at the risk of inducing in the reader a sense of rigour which in fact does not exist on the physical problem. As a concept, it allows a correct definition of averaging. But that averaging can only be realized on a mental experience. So, on the scope of this course we discard it and use, instead, the experimentaly measurable mean quantity defined by the following relations. These are only rigorous on the limit of $\Delta t = \infty$ However $\Delta t = \infty$ would surpress all unsteady turbulent flows from the discussion. For $\Delta t$ finite, the approach is to consider it as a typical «time scale» of the flow under consideration. This, again, is a typical dubious definition of present day turbulence theory ...

we can write 23) as:

$$24)\quad \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu \frac{\partial U_i}{\partial x_j} - \overline{u_i u_j}\right]$$

which are similar to the equations for purely laminar flow except for the term $\overline{u_i u_j}$. This new term is not known because we have introduced $u_i$ as an aditional unknown in the starting equations. This aditional unknown which transform the system on an undeterminated one is the «price» for the mathematical regularization got, and it translates the loss of information introduced by the averaging process.

The term $\overline{u_i u_j} = R_{ij}$ is an analogue of the viscous stress tensor and is called the Reynolds tensor (disregarding a constant factor). In fact, regarding the mean flow , $R_{ij}$ is a momentum flux due to the random motions of the fluid, analogue of the randon motion of molecules which account for the laminar viscosity. This analogy is, however, an overview because $R_{ij}$ expresses a macroscopic motion, which must obey the N.S. equations. As a first aproximation, started by Boussinesq, we can go further with the analogy and postulate

$$R = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

thereby introducing the concept of a turbulent viscosity $\mu_t$.(*)

Even as a crude assumption, $\mu_t$ has many advantages from an engineering point of view and provided its limitations are known, this approach is one of the most sucessfull presently, even when a crudest assumption is made regarding $\mu_t$ as in Prandtl mixing-length theory, to be discussed later.

However, before we simplify the problem for engineering sake we must further explore the averaging process on the basic equations in order to get a better insight as to postulate $R_{ij}$.

The next step is to introduce the averaging process in the kinetic energy equation.

## 4 – THE TURBULENT KINETIC ENERGY EQUATION

### 4.1 Introducing

$$V = U_i + u_i$$

$$P^* = P + p$$

in the kinetic energy equation 20) and taking into account

$$\overline{V_i V_i} = \overline{(U_i + u_i)^2} = \overline{U_i^2 + u_i^2}$$

$$\overline{V_i V_i V_i} = \overline{(U_i + u_i)(U_i + u_i)^2}$$

$$= U_i U_i U_i + \overline{u_i u_i u_i} + 2\,\overline{u_i u_j}\, U_i$$

we get

$$24)\quad \frac{\partial}{\partial t}\left(\frac{1}{2}\rho U_i U_i\right) + \frac{\partial}{\partial x_i}\left[\rho U_j\left(\frac{1}{2}U_i U_i + \frac{P}{\rho}\right) - \frac{1}{2}\mu\frac{\partial (U_i U_i)}{\partial x_j}\right] + \mu\frac{\partial U_i}{\partial x_j}\cdot\frac{\partial U_i}{\partial x_j} +$$

$$+ \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\,\overline{u_i u_i}\right) + \frac{\partial}{\partial x_i}\left[\rho u_j\left(\frac{1}{2}u_i u_i + \frac{p}{\rho}\right) - \frac{1}{2}\mu\frac{\partial \overline{u_i u_i}}{\partial x_j}\right] + \mu\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_i}{\partial x_j} +$$

$$+ \frac{\partial}{\partial x_j}\left(\rho\,\overline{u_i u_j}\, U_i + \frac{1}{2}\rho\,\overline{U_i^2}\, U_j\right) = 0$$

If we compare 24) with the starting equation 20) we note that the first group of terms in 24) corresponds exactly to the kinetic energy of a flow with velocity $U_i$, and the second group has the same meaning, in the mean, for a flow of velocity $u_i$.

(*) $\mu_t$ is always considered as a scalar quantity which is physically unacceptable. That would imply $\overline{R_{ij}} = 0$ if the mean flow as no gradient. More than that, $\overline{R_i} = \overline{u_i^2}$ is proporcional on the l.h.s. to the mean kinetic energy, but $\overline{u_i^2} = \mu\frac{\partial U_i}{\partial x_i} = 0$ if the flow is incompressible. This would imply that an incompressible fluid would not have turbulence.

With the Boussinesq approximation and $\mu$ a scalar, the assumption is, at best, a usefull approximation.

However, when the two flows are added, due to the nonlinearity, a new group of terms arise:

$$\frac{\partial}{\partial x_j}\left[\overline{\rho u_i u_j} U_i + \frac{1}{2}\overline{\rho u_i^2} U_j\right]$$

This group is just a consequence of separating the instantaneous velocity field in two by

$$V_i = U_i + u_i$$

with the aditional condition $\overline{u_i} = 0$

This means that 24) is exact as long as the N. S. equations are, and the assumptions regarding continuity of the fields are verified,

If we wish to separate 24) in the kinetic energy of the mean motion and of the turbulent field, conceptual consistency with the definition $\overline{u_i} = 0$ implies:

— The kinetic energy of the mean flow must result from the basic equations of the mean flow.

— The kinetic energy of the turbulence must result from the equations for $u_i$ (before averaging).

— The sum of the kinetic energy of both flows must be given by 24).

For the mean flow we have, from 23)

$$25)\quad \rho\left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right] = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j \, d x_j} - \frac{\partial}{\partial x_j}\overline{\rho u_i u_j}$$

$$\frac{\partial U_j}{\partial x_j} = 0$$

and by diference to.22)

$$26)\quad \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\overline{\rho u_i u_j}$$

$$\frac{\partial u_j}{\partial x_j} = 0$$

From 25) we have already seen that relative to a laminar flow we have the additional term $\frac{\partial}{\partial x_j}(\rho u_i u_j)$ (the divergence of the Reynolds stress tensor). When we multiply 25) by $U_i$, relative to a laminar flow arises a new term in the equation of the kinetic energy due to the work done by the Reynolds stresses:

$$27)\quad U_i \frac{\partial}{\partial x_j}\overline{\rho u_i u_j} = \frac{\partial}{\partial x_j}\left[\overline{\rho u_i u_j}\, U_i\right] - \overline{\rho u_i u_j}\frac{\partial U_i}{\partial x_j}$$

If we now compare 26) with the equations for a laminar flow of velocity $u_i$ we see that the momentum balance of the total flow $V_i = U_i + u_i$ and the definition of mean flow gave rise to the new terms:

$$28)\quad \rho u_j\left[\frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j}\right] - \frac{\partial}{\partial x_j}\overline{\rho u_i u_j}$$

The last one is not «time dependent» (on the scale of averaging) and is just the divergence of the Reynolds stress tensor with the sign changed.

Multiplying 26) by $u_i$ and averaging the first two terms in 28) give rise to the energy terms:

$$29)\quad \overline{\rho u_i u_j}\frac{\partial U_i}{\partial x_j} + \rho U_j \overline{u_i \frac{\partial u_i}{\partial x_j}}$$

The last one can be written

$$\rho U_j \overline{u_i \frac{\partial u_i}{\partial x_j}} = U_j \frac{\partial}{\partial x_j}\frac{1}{2}\overline{\rho u_i u_i}$$

which is easily seen to be a convective transport of $\frac{1}{2}\overline{\rho u_i^2}$ by the mean flow.

Coming back to 24) we see that the terms on the last line have already appeared in 27) and 29).

However, the term $-\overline{\rho u_i u_j}\frac{\partial U_i}{\partial x_j}$ in the equation of the kinetic energy for the mean flow, and the term $\overline{\rho u_i u_j}\frac{\partial U_i}{\partial x_j}$ in the equation of kinetic energy of the $u_i$ flow do not appear in equation 24) of the kinetic energy of the total flow. These terms cancel out when both the kinetic energies are added and this clearly explains its meaning:

— they represent an energy transfer betwen the mean flow and the $u_i$ flow.

Because de $u_i$ flow represents the turbulent field, we call the equation of its kinetic energy the turbulent kinetic energy equation.

$$\text{Putting } \overline{\frac{1}{2}\rho u_i u_i} = \overline{q^2}$$

we can write the equation as:

$$\frac{\partial}{\partial t} q^2 + U_j \frac{\partial q^2}{d x_j} = -$$

$$- \frac{\partial}{\partial x_j}\left[ \overline{u_j\left(\frac{1}{2} q^2 + p\right)} - \mu \frac{\partial q^2}{\partial x_j} \right] -$$

$$30) \qquad -\rho \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} - \mu \left( \frac{\partial u_i}{\partial x_j} \right)^2$$

Except for the terms $U_j \frac{\partial q^2}{\partial x_j}$ and $-\rho \overline{u_i u_j} \frac{\partial u_i}{\partial x_j}$ the equation is exactly the same as for the laminar flow, and as such the terms have a completely similar meaning. A more familiar interpretation, however, is possible if we remark that the aditional term $U_j \frac{\partial q^2}{\partial x_j}$ is just a convection by the mean flow of the scalar quantity $q^2$ This allows writing the 1. h. s. of 30) as

$$\frac{d q^2}{d t}$$

On the r. h. s. the meaning of the new term $\rho \overline{u_i u_j} \frac{\partial U_i}{\partial x_j}$ which is a source term has already been explained — it is a source term due to the transfer of energy between the mean flow and the turbulent field.

The last term: $\mu \left( \frac{\partial u_i}{\partial x_j} \right)^2$ is the viscous dissipation which is responsable for the conversion of the kinetic energy to internal energy*.

4.2. In the usual texts the turbulence kinetic energy equation is derived without reference to the total kinetic energy. However, we must stress the fact that the basic equations are the N. S. time dependent equations and the kinetic energy of the total flow. This will avoid any misinterpretation regarding the physical meaning of some terms like the dissipation. When the splitting between a mean and a intantaneous flow is done new terms arise. There are terms to which a physical meaning can be atributed. For instance, a term equivalent to the viscous stress tensor-which gives rise to the equivalent of a dissipation, which is not a true dissipation because the Reynolds stresses are not viscous stresses. The analogy, however, is a very helpefull one provided we are aware of its limitations.

Because the use of vectorial identities and the continuity equation allows the equations to be writen in diferent form, the exact form of each term will not always be the same neither the physical meaning which we give to them. These must be born in mind when comparing diferent texts of different authors.

We will present now a common form in many texts of the turbulence kinetic energy equation. From equation 26) if we write

$$k = \frac{1}{2} \overline{u_i u_i} = \frac{1}{2} \Sigma u_i^2 \quad (q^2 = \rho k^2)$$

$$31) \qquad \frac{\partial k}{\partial t} + \overline{u_j k} + \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} +$$

$$+ U_j \frac{\partial k}{\partial x_j} = - \frac{1}{2} u_i \frac{\partial p}{\partial x_i} + \nu u_i \overline{\frac{\partial^2 u_i}{\partial x_j^2}}$$

where the interpretation of the terms is:

$$32) \qquad U_j \frac{\partial}{\partial x_j} \overline{u_i^2} = U_j \frac{\partial}{\partial x_j} k^2 =$$

---

* There are some confusions regarding dissipation Hinze says (p. 66) that $\left( \frac{\partial u_i}{\partial x_j} \right)^2$ is not a dissipation which can only be due to typographical error.

In fact $\mu \left( \frac{\partial u_i}{\partial x_j} \right)^2$ is the only intrinsically positive term on the kinetic energy equation. Besides, in the absence of mean flow, integrating on a closed volume which goes to $\infty$ or is bounded by closed walls (where $U_i = 0$) gives

$$\frac{d}{d t} \int q^2 d v = - \mu \int \left( \frac{\partial u_i}{\partial x_j} \right)^2 d v$$

If $\mu \left( \frac{\partial u_i}{\partial x_j} \right)^2$ where not a dissipation no other terms could be and the kinetic energy of turbulence would be conserved for ever.

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$$= U_1 \frac{\partial k^2}{\partial x_1} + U_2 \frac{\partial k^2}{\partial x_2} + U_3 \frac{\partial k^3}{\partial x_3}$$

This terms represent the convection or advection of $k^2$ by the mean flow.

All the terms can be experimentaly found

33)
$$\overline{u_i u_j} \frac{\partial U_i}{\partial x_j} = \boxed{\overline{u_1 u_1} \frac{\partial U_1}{\partial x_1} + \overline{u_1 u_2} \frac{\partial U_1}{\partial x_2} + \overline{u_2 u_1} \frac{\partial U_2}{\partial x_1} + \overline{u_2 u_2} \frac{\partial U_2}{\partial x_2}} + \overline{u_1 u_3} \frac{\partial U_1}{\partial x_3} + \overline{u_2 u_3} \frac{\partial U_3}{\partial x_3} + \overline{u_3 u_1} \frac{\partial U_3}{\partial x_1} + \overline{u_3 u_2} \frac{\partial U_3}{\partial x_2} + \overline{u_3 u_3} \frac{\partial U_3}{\partial x_3}$$

These terms are source terms and represent the «production» of turbulence energy from the mean flow, as already explained. This means a transfer of energy from one flow to the other.

If the mean flow has no gradients (U = const.) there is no transfer of energy to the turbulence. So, unless turbulence is diffused or convected from other regions of the flow, the kinetic energy of turbulence vanishes due to the dissipation.

If the flow is two-dimensional 33) reduces to the four terms enclosed by the dashed square: these terms will be discusssed later in connection with elliptic flows.

All the terms in 33) can be found experimentaly.

34)
$$\frac{\partial}{\partial x_j} \overline{u_i^2 u_j} = \frac{\partial}{\partial x_j} \overline{u_j k^2}$$
$$= - \frac{\partial}{\partial x_1} \overline{(u_1 k^2)} + \frac{\partial}{\partial x_2} \overline{(u_2 k^2)} + \frac{\partial}{\partial x_3} \overline{(u_3 k^2)}$$

These terms have the meaning of a difusion of $k^2$ by the turbulent motion itself. They are very difficult to measure experimentaly

35)
$$\overline{\frac{1}{\rho} u_i \frac{\partial p}{\partial x_i}} =$$
$$= - \frac{1}{\rho} \left[ \overline{u_1 \frac{\partial p}{\partial x_1}} + \overline{u_2 \frac{\partial p}{\partial x_2}} + \overline{u_3 \frac{\partial p}{\partial x_3}} \right]$$

Is also the equivalent of a difusion due to the coupling betwen pressure and velocity flutuations.

Impossible to measure directly with presently known measuring techniques.

36)
$$\nu \overline{u_i \frac{\partial u_i}{\partial x_j^2}} = 2 \nu \frac{\partial^2}{\partial x_j^2} (k^2) - \nu \overline{\left( \frac{\partial u_i}{\partial x_j} \right)^2}$$

The first term, $\nu \frac{\partial^2}{\partial x_j^2} (k^2)$ is difusion of k due to its own gradient, like heat difuses due to a temperature gradient and termal conductivity or momentum due to viscosity and velocity gradient.

The second term:

$$\nu \overline{\left( \frac{\partial u_i}{\partial x_j} \right)^2} = \nu \overline{\left( \frac{\partial u_1}{\partial x_1} \right)^2} + \overline{\left( \frac{\partial u_1}{\partial x_2} \right)^2} + \overline{\left( \frac{\partial u_1}{\partial x_3} \right)^2} + \overline{\left( \frac{\partial u_2}{\partial x_1} \right)^2} + \overline{\left( \frac{\partial u_2}{\partial x_2} \right)^2} + \overline{\left( \frac{\partial u_2}{\partial x_3} \right)^2} + \overline{\left( \frac{\partial u_3}{\partial x_1} \right)^2} + \overline{\left( \frac{\partial u_3}{\partial x_2} \right)^2} + \overline{\left( \frac{\partial u_3}{\partial x_3} \right)^2}$$

represents the viscous dissipation of turbulence, as already discussed.

Only the terms $\left( \frac{\partial u_i}{\partial x_i} \right)^2$ are measured with not to much dificulty.

Usually, the terms 32) and $\nu \frac{\partial^2}{\partial x_j^2} (k^2)$ are grouped under the heading of «difusion terms» and its value acessed experimentaly by diference in order to close the kinetic energy balance.

## NOMENCLATURE

$E$ — Total kinetic energy per unit volume
$F$ — External force per unit mass
$q^2$ — Turbulent kinetic energy per unit volume
$k$ — Turbulent kinetic energy per unit mass $\left(\frac{1}{2} u_i u_i\right)$
$J_\phi$ — Flux density of $\phi$
$Q_\phi$ — Source of $\phi$ per unit mass
$U_i$ — Mean velocity component in direction i
$u_i$ — Flutuating velocity component in direction i
$V_i$ — Velocity component in direction i
$x_i, x_j$ $(i, j = 1, 2, 3)$ orthogonal coordinates
$\tau_{ij}$ — Total stress tensor
$\phi$ — Any physical property per unit mass
$\mu$ — Dynamic viscosity
$\nu$ — Kinematic viscosity
$\rho$ — Specific mass

*Obs:* — An overbar (i. e $\overline{u_i u_j}$) means always a time average — Repeated indices means summation on corresponding terms (Einstein convention; i. e $V_j \frac{\partial V_i}{\partial x_j} \equiv \sum_j V_j \frac{\partial V_i}{\partial x_j}$;

$$U_i^2 \equiv \sum_{i=1}^{3} U_i U_i \; ; \; \frac{\partial^2 U_i}{\partial x_j^2} \equiv \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \cdot \frac{\partial}{\partial x_j} U_i)$$

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# II — PREDICTION PROCEDURES FOR TWO — DIMENSIONAL, BOUNDARY LAYER FLOWS: EQUATIONS AND ASSUMPTIONS

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RESUMO

*Consideram-se e discutem-se as hipóteses e equações relevantes para métodos de cálculo dos escoamentos médios bidimensionais do tipo camada limite em regime turbulento.*

*Formulam-se e discutem-se os tipos de equações que descrevem as características de turbulência, desde o modelo simples que não utiliza equações diferenciais, até aos mais complexos que envolvem sistemas de equações simultâneas. As vantagens relativas dos vários tratamentos são apresentadas. A justificação física para as hipóteses inerentes em cada conjunto de equações é indicada.*

SYNOPSIS

*The equations and assumptions relevant to prediction procedures for turbulent, two-dimensional, boundary--layer flows are considered and discussed. Sets of equations, from the simple case where no differential equations are used to describe the turbulence characteristics, to the more complex cases which require several differential equations, are formulated and the relative advantages of each set is made clear. The physical justification for the various assumptions in each set of equations is indicated.*

## 1. INTRODUCTION

The subject of the paper is clearly stated in the title. Prediction procedures of varying degrees of complexity will be synthesised using forms of the equations derived in the earlier paper of J. J. D. Domingos (1), together with physical reasoning. It is necessary to stress the words «prediction procedures». The subject of turbulence models is *interesting* for its own sake because it taxes our mental ingenuity and mathematical skill; it is *useful* in so far as it enables us to predict important flow properties which would otherwise have to be measured. In almost all cases measurements are expensive in time and money; the purpose of a prediction procedure is to replace expensive measurements by cheap, but accurate, predictions.

Attention must also be drawn to the restrictions mentioned in the title. The reason for the restriction of *two-dimensional flow* was mentioned in ref. 1; the solution of the three-dimensional equations requires more storage than is available in the present generation of computers. *Boundary layer flows* are considered because the equations which govern such flows are a reduced form of those which govern recirculating flows. Thus, the equations discussed in this paper will be the parabolic, boundary layer equations rather than the elliptic steady Navier-Stokes equations. It should be remembered, however, that a very large class of flow situations is governed by boundary-layer equations; the classe includes jets, wakes, mixing regions, wall wakes and wall jets, pipe flow, diffuser flow, etc. The subject of elliptic flows will be considered by J. J. D. Domingos in a later paper.

In the subsequent sections, soluble sets of equations will be synthesised to represent turbulent, two-dimensional boundary layer flow. Assumptions are necessary to obtain these soluble sets of equations and will be discussed in some detail. The relative advantages and disadvantages of the various sets will be made clear. Specific predictions will not be presented in this

paper but reference to sources of such predictions will be quoted.

The five sets of equations to be considered here are noted in Table 1 and are classified according to the number of differential equations used to characterise the turbulent nature of the flow. By not specifying in Table 1 a number of equations for the turbulent flow greater than 3, it is implied that there is a law of diminishing return; this point will be clarified later in the paper.

Of course, where heat and mass transfer are important, the equations for total enthalpy and species concentration must also be solved. The content of the present paper is restricted to the aerodynamic equations but a similar treatment

TABLE 1

| Number of differential equations to characterise turbulence | Dependent variables of differential equations | Auxiliary equations | References | Comment |
|---|---|---|---|---|
| 0 | 0 | $\tau = \rho l^2 \left\lvert \frac{\partial U}{\partial y} \right\rvert \frac{\partial U}{\partial y}$ | 2 | Simple and tractable: well tested: mixing length is surprisingly general: no account of turbulence. |
| | | $\mu_t = c \rho \delta_1 U_G$ | 3,4 | |
| | | etc. | | |
| 1 | $\nu_t$ | $-\overline{uv} = \nu_t \frac{\partial U}{\partial y}$ | 16 | Influence of turbulence intensity is considered: no influence of scale: not universal. |
| | k | $-\overline{uv} = c_\mu l k^{\frac{1}{2}}$ | 18,23,24 | |
| | $\tau$ | $-\overline{uv} = A k$ | 17 | |
| 2 | $k, kl$ | $\mu_t = c_\mu l_\mu \rho k^{\frac{1}{2}}$ | 23 | Signigicantly greater universality: significant increase in complexibility: different zero locations of $\overline{uv}$ and $\partial U/\partial y$ not predicted. |
| | $k, k/l^2$ | $\mu_t = c_\mu l_\mu \rho k^{\frac{1}{2}}$ | | |
| | $k, S$ | $\mu_t = c_\mu (\rho k)^2 / S$ | 19,21,22 | |
| | $k, w$ | $\mu_t = c_\mu \rho k w^{\frac{1}{2}}$ | 24 | |
| 3 | $k, kl, \overline{uv}$ | NONE | 26 | $\overline{uv}$ and $\partial U/\partial y$ can now become zero at different locations: increased complexity. |
| | $k, S, \overline{uv}$ | | 25 | |
| MORE | 4 equations<br>23 equations | | 27<br>28 | The greater the number of equations, the greater the universality. The nature of the necessary assumptions does not warrant this degree of complexity for boundary-layer flows. |

is possible for the total enthalpy or species concentration; the present absence of reliable measurements, e. g. for $\overline{vt}$, tends to render such an approach unjustifiable.

As a final comment in this introduction, it is important to realise that, for present design purposes, only eddy viscosity models (zero differential equations for the turbulence) have been tested with sufficient care and over a sufficiently wide range. For instance, the mixing-length hypothesis, together with the computational procedure of D. B. Spalding and S. V. Patankar (2), has been tested over a very wide range of flow configurations (*) and its successes and failures are now well known: this cannot be said for the higher order turbulence models. It is clear, however, that models such as the mixing length have defects and higher order models are necessary to remove the defects. Thus, the design procedures of the future are likely to utilise two or threee equations for the turbulent flow.

## 2. ZERO EQUATION TURBULENCE MODELS

In this case, the problem is to solve the mean flow equation:

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial}{\partial x_j}\overline{u_i u_j} \tag{1}$$

in its boundary-layer form,

$$U \frac{\partial U}{\partial y} + V \frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dP}{dx} + \nu \frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}\overline{uv} \tag{2}$$

together with the continuity equation

$$\frac{\partial U_i}{\partial x_i} = 0 \quad . \tag{3}$$

For practical purposes, the velocity profile development, $U = U(x, y)$, and the value of the wall shear stress, are the dependent variables of greatest interest. For laminar flow the last term in equation 2, the Reynolds stress term, is zero and similar or numerical solutions can be found provided boundary conditions are specified at

$$x = 0 \quad , \quad y = 0 \quad \text{and} \quad y \longrightarrow \infty \; .$$

The equation is parabolic, as are almost all equations considered here, so marching solution procedures are possible. In practice the stream function or non-dimensional stream function is used to satisfy equation 3 and eliminate V from equation 2; the resulting equation remains parabolic.

For high Reynolds number turbulent flow, the Reynolds stress term of equation 2 is very much larger than the viscous stress term. By analogy with the viscous stress term, it may be assumed to have the form:

$$-\frac{\partial}{\partial y}\overline{uv} = \frac{\partial}{\partial y}\left(\nu_t \frac{\partial U}{\partial y}\right) \quad . \tag{4}$$

The set of equations comprising 2, 3 and 4 is soluble provided $\nu_t$ is specified in terms of the dependent or independent variables.

Many eddy-viscosity hypotheses have been proposed and used, in conjunction with numerical procedures and further assumptions for the near-wall region, to predict mean-velocity profiles and wall-shear stress. Both G. Mellor (3) and A. M. O. Smith and T. Cebeci (4) who are concerned principally with boundary layers on adiabatic plane walls, employ an eddy-viscosity hypothesis which, in the outer region of the flow, has the form:

$$\nu_t = C \delta_1 U_G$$

where $\delta_1$ (x) is the displacement thickness of the boundary layer.

This particular hypothesis works well for the range of flows for which it was intended; it is inappropriate, however, to the large class of flows which are characterised by maxima in their mean velocity profile. Clearly, therefore, this hypothesis is restricted in its applicability.

A more general eddy-viscosity hypothesis is that based on Prandtl's mixing length, i.e.

$$\nu_t = l^2 \left| \frac{\partial U}{\partial y} \right|$$

or

$$-\overline{uv} = l^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y} \tag{5}$$

where the mixing length, $l$, must be specified. This hypothesis with $l$ specified by

(*) This procedure was the subject of an extension course entitled «Heat and Mass Transfer in Turbulent Boundary Layers» at Instituto Superior Técnico in July 1969.

$$l = 0.435\,y \quad , \quad 0 < y < \frac{0.09}{0.435}\,y_G$$

$$l = 0.09\,y_G \quad , \quad \frac{0.09}{0.435}\,y_G < y \quad , \qquad (6)$$

has been widely tested at Imperial College and was discussed in the Extension Course referred to on page 301. In that course, it was shown that successful predictions of important dependent variables were possible over a wide range of dependent variables and for many pratically important situations. Nevertheless, it was clear that the empirical specification of mixing length and the hypothesis itself were not universal: for example, the constant 0 09 did not permit the correct prediction of both radial and plane wall--jet flows.

The above examples of eddy-viscosity and mixing-length hypotheses were presented to make the point that both are valid over limited ranges of flow — the mixing-length over a surprisingly wide and useful range — but that both are far from universal descriptions of the mean motion of a turbulent flow. Consider also the following more general statements of the inadequacy of such models:

1. Shear stress depends upon *local* mean velocity. But do upstream and downstream properties influence $\overline{u_i u_j}$?
2. Shear stress is zero when $\partial U/\partial y$ is zero. Is this in accord with experimental evidence?
3. Flow properties are uninfluenced by the scale and intensity of the turbulence. Can this be true?

It is not true that the local shear stress depends only upon the local mean velocity or its gradient. The upstream history effects in boundary layer flows are wellknown and have been commented upon in ref. 5. On the other hand, the nature of parabolic differential equations does not allow the downstream properties influence the boundary layer flow.

In strongly asymmetric flow, it is known, that the zero shear stress and zero local mean velocity gradient do not occur at the same location. For instance, in wall-jet flows, it has been shown that the zero shear stress position is located closer to the wall than is the zero local mean velocity gradient (6, 7). Also, in plane ducts where one wall is rough and the other smooth, the zero shear stress location lies closer to the smooth wall than does the zero mean-velocity-gradient location (8). Finally, and most important, it is well-known that the scale and intensity of turbulence can significantly influence mean flow properties of turbulent boundary layers in certain situations. Examples which may be cited include the very substantial increase in heat transfer rate caused by a small increase in free-stream turbulence close to the stagnation point of a cylinder (9); the enhanced mixing of a mixing region caused by an increase in free-stream turbulence (10); and the lower transitional Reynolds numbers which are accompanied by the higher turbulence intensity levels (11).

The arguments which led Prandtl to propose his mixing-length hypothesis are well-documented (12) and need not be repeated here. They imply, however, that any dissipation of turbulent energy is instantaneously countered by a production of an equal amount. This, in turn, implies that diffusion and convection of turbulent energy are absent. But experiments have shown that this is only true in the logarithmic region of a wall boundary-layer flow (12). This represents a further defect of the mixing-length model.

An interesting exercise (13) which may be performed in connection with eddy-viscosity hypothesis is to differentiate the general form of the eddy-viscosity hypothesis,

$$-\overline{u_i u_j} = \nu_t \left[ \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] \quad , \qquad (7)$$

with respect to $x_k$ and multiply the result by $U_k$.

This will yield a shear stress equation which may be compared with that derivable from the Navier-Stokes equations without the eddy--viscosity hypothesis. For the two equations to be equivalent, certain terms must be equal. The reader should consider for himself whether this circumstance is likely.

The broad conclusion which may be drawn from the above discussion and from applications of eddy-viscosity hypotheses* is that although they have a limited range of applicability, the

* The reader is referred to reference 15 for details of some further successes and failures of eddy-viscosity hypotheses.

range is sufficiently wide to justify their use in many cases. In particular, the mixing-length hypothesis has been shown (14) to permit satisfactory predictions of practically important dependent variables over a wide range of flow situations. Eddy-viscosity hypotheses are easily understood, relatively simple to use and, since they require the solution of only one differential equation, comparatively economical of computer time.

## 3. ONE-EQUATION TURBULENCE MODELS

In devising a turbulence model which requires the solution of one differential equation for the turbulent flow, perhaps the simplest and most obvious approach is that of L. S. G. Kovasznay and V. N. Nee (16). In this case, the authors solved equations 2 and 3 together with equation 4 and a differential equation for $\nu_t$. The differential equation for $\nu^t$ was not derived from first principles but was written down in the form given by equation 8:

$$\underbrace{U\frac{\partial \nu_t}{\partial x} + V\frac{\partial \nu_t}{\partial y}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial y}\left(\nu t\frac{\partial \nu_t}{\partial y}\right)}_{\text{diffusion}} + \underbrace{(A\,\nu_t - \nu)\frac{\partial U}{\partial y}}_{\text{production}} - \underbrace{\frac{B}{L^2}\,\nu_t\,(\nu_t - \nu)}_{\text{dissipation}} \tag{8}$$

The form of this equation is similar to that of other equations which may be derived for the turbulent flow (as will be shown later) and this was the part justification for the inclusion of four terms. The coefficients A and B are hopefully universal constants and L is a length scale which must be specified empirically. The equation implies that the eddy viscosity is convected, diffused, produced and dissipated in a manner, for example, similar to turbulent kinetic energy. The criticisms levied at eddy-viscosity models also apply to this one but, since there are more constants to adjust, it is likely that a slightly greater degree of universality may be attained.

The more useful one-turbulence equation models are those which include the solution of the turbulent kinetic energy equation in addition to mean momentum and continuity. The two dimensional form of the kinetic energy equation, without assumptions, may be written as:

$$U_j\frac{\partial}{\partial x_j}\overline{u_i^2} + \overline{2u_iu_j}\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u_i^2u_j} = 2\,\overline{\frac{u_i}{\rho}\frac{\partial P}{\partial x_i}} + 2\nu\left(\overline{u_i\frac{\partial^2 u_i}{\partial x_j^2}}\right) \tag{9}$$

The inclusion of equation (9) permits the mean flow to be influenced by the turbulent energy but the deficiencies 1 and 2 of the eddy-viscosity model still apply. The terms in equation (9) have been discussed in some detail in ref. 1 and more will be said, regarding their measurement, in a subsequent paper. The two-dimensional, boundary-layer form of equation (9) may be written as:

$$\underbrace{U\frac{\partial}{\partial x}k + V\frac{\partial}{\partial y}k}_{\text{convection}} + \underbrace{\overline{(u^2 - v^2)}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y}}_{\text{production}} + \underbrace{\frac{\partial}{\partial y}\overline{v(k + p/\rho)} - \nu\frac{\partial^2}{\partial y^2}\overline{(k + v^2)}}_{\text{diffusion}} + \underbrace{\epsilon}_{\text{dissipation}} = 0 \tag{10}$$

It should be clear that the set of equations comprising continuity, mean momentum and equation (10) cannot be solved without assumptions. First, a link must be provided between $\overline{uv}$ and k. One such link is that used by P. Bradshaw (17) i.e. $\overline{uv} \propto k$; another is the Prandtl--Kolmogorov formulation used extensively at Imperial College, i.e. $\overline{uv} \propto k^{\frac{1}{2}}\,\partial U/dy$, or $\nu_t = C_\mu l_\mu \rho k^{\frac{1}{2}}$. In addition, the diffusion and dissipation terms are intractible in their present form. For high turbulent Reynolds numbers, the viscous dissipation may reasonably be neglected but the remaining component of diffusion remains and includes both v and p about which our set of equations provides no information. Indeed, the fluctuating pressure represents a substantially unknown quantity since it has not been measured except at a wall. The term is, therefore, approximated by:

$$\frac{\partial}{\partial y}\left[\frac{\nu_t}{\sigma_{k,t}}\frac{\partial k}{\partial y}\right].$$

There is little direct evidence of the validity of this assumption which implies the existence of a Prandtl number for turbulent energy and requires that the kinetic energy diffuses down its own gradient. On the other hand, other scalar quantities such as enthalpy or mass concentration behave in this way so, in the absence of alternative information, it is a plausible assumption. Of course, the plausibility decreases as eddy size increases. With this assumption, we must continually ask the question «is $\sigma_{k,t}$ a constant or does it vary from one flow situation to another, or from one part of the flow to another?» Tests carried out at Imperial College suggest that $\sigma_{k,t}$ is constant over a very wide range of flow situations. Further testing is, however, necessary to confirm the actual range over which it may be assumed constant.

The dissipation term, $\epsilon$, must also be approximated. It is defined as:

$$\epsilon \equiv \nu \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} \quad . \tag{11}$$

The nine components of $\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2}$ cannot be measured and cannot be included, per se, in a one-equation model. It is assumed, therefore, that energy is transferred from larger to small eddies (a cascade effect) until there are no small eddies to which the energy may be transferred. The very small eddies are assumed to lose energy as a result of viscous stresses. So dissipation takes place at high wave numbers (i. e high values of $2\pi f/U$) and in a region of the energy spectrum where it can be argued that the turbulence is isotropic. With this assumption it can be shown that:

$$\epsilon \equiv \frac{C_D k^{3/2}}{l_D} \tag{12}$$

where $C_D$ is a constant and $l_D$ is a length scale characteristic of the size of the larger eddy. Thus the diffusion and dissipation terms equation (9) may be rewritten as:

$$U \frac{\partial}{\partial x} k + V \frac{\partial}{\partial y} k + \overline{(u^2 - v^2)} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial U}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_{k,t}} \frac{\partial k}{\partial y}\right) + \frac{C_D k^{3/2}}{l_D} = 0 \tag{13}$$

For boundary layer flows, the production by normal stresses can usually be neglected and, if this is so, the set of equations comprising continuity, mean momentum and turbulent kinetic energy is soluble provided that a turbulent viscosity law is specified. D. B. Spalding (18) strongly favours the Prandtl-Kolmogorov viscosity law in the form:

$$\nu_t = c_\mu l_\mu k^{\frac{1}{2}} \tag{14}$$

This set of equations requires a specification of $\sigma_{k,t}$, $C_D$, $l_D$, C and $l$ and these must be found from experimental information. Because dissipation and production are known to be equal in the logarithmic region close to the wall, the law of the wall provides a link between $C_D$ and $C_\mu$, i. e.

$$\frac{C_\mu^{\frac{3}{4}}}{C_D^{\frac{1}{4}}} \frac{U}{(\tau_W/\rho)^{1/2}} = \ln\left(\frac{E U \sqrt{\tau_W \rho}}{\mu}\right) \quad . \tag{15}$$

Thus, a choice of $C_\mu$ implies a choice of $C_D$ and only four parameters need to be specified.

Bradshaw's approach favours a viscositylaw linking $\overline{uv}$ with k only; this allows the kinetic energy equation to be transformed to an equation with shear stress as dependent variable. In this shear stress equation, the diffusion term is treated in a different manner from that indicated above and the resulting set of equations turn out to be hyperbolic; they are then solved by the method of characterisitics.

There is little doubt that the one-equation models are superior to zero equation models from the point of view of generality; the larger number of assignable constants ensures this. Bradshaw's method has been shown to allow very satisfactory predictions of mean velocity and wall shear stress over a very wide range of boundary layer flows; the proportionality between shear stress and kinetic energy ensures, however, that the procedure will not work, in its present form, for flows with velocity maxima. On the other hand, the procedure was not deve-

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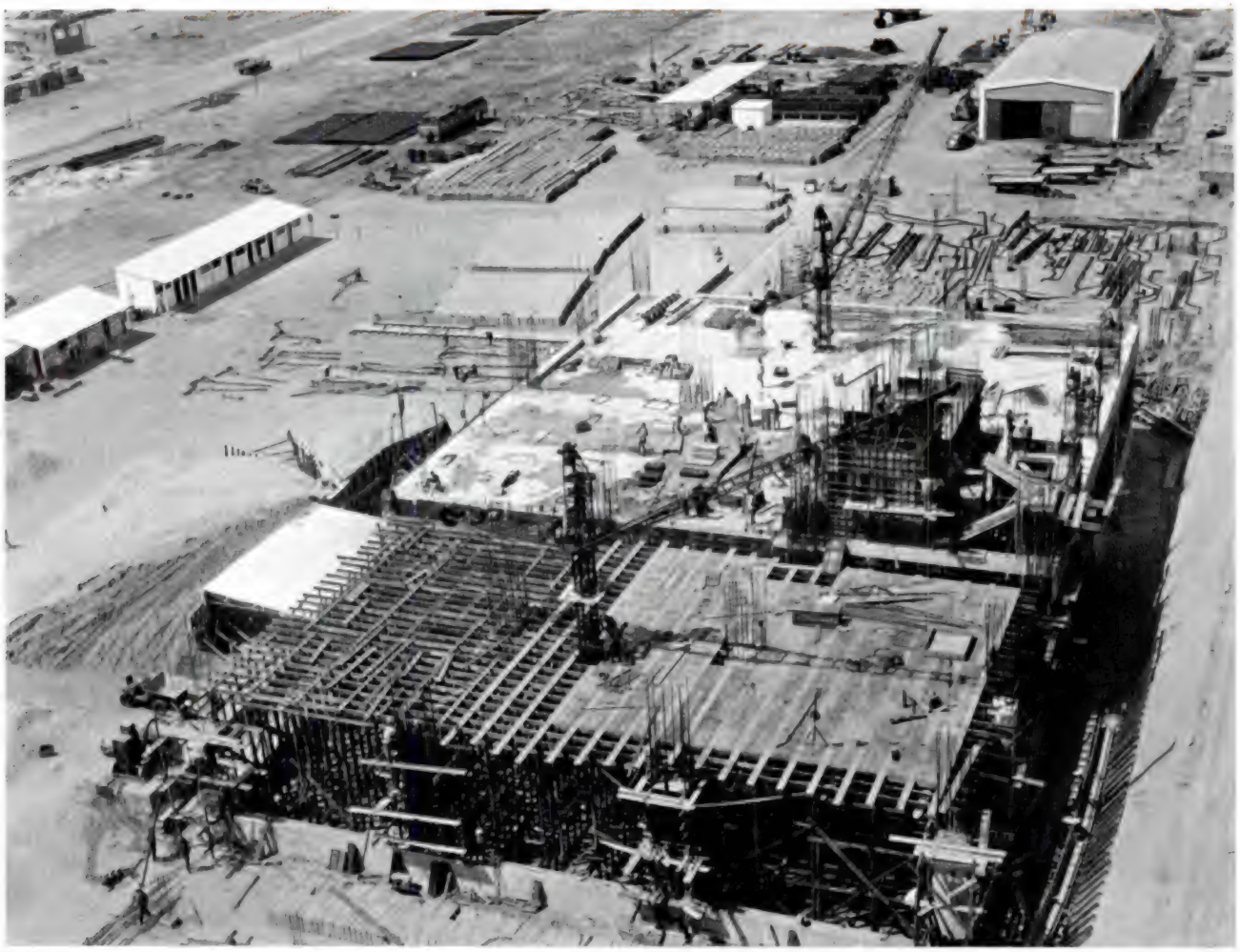

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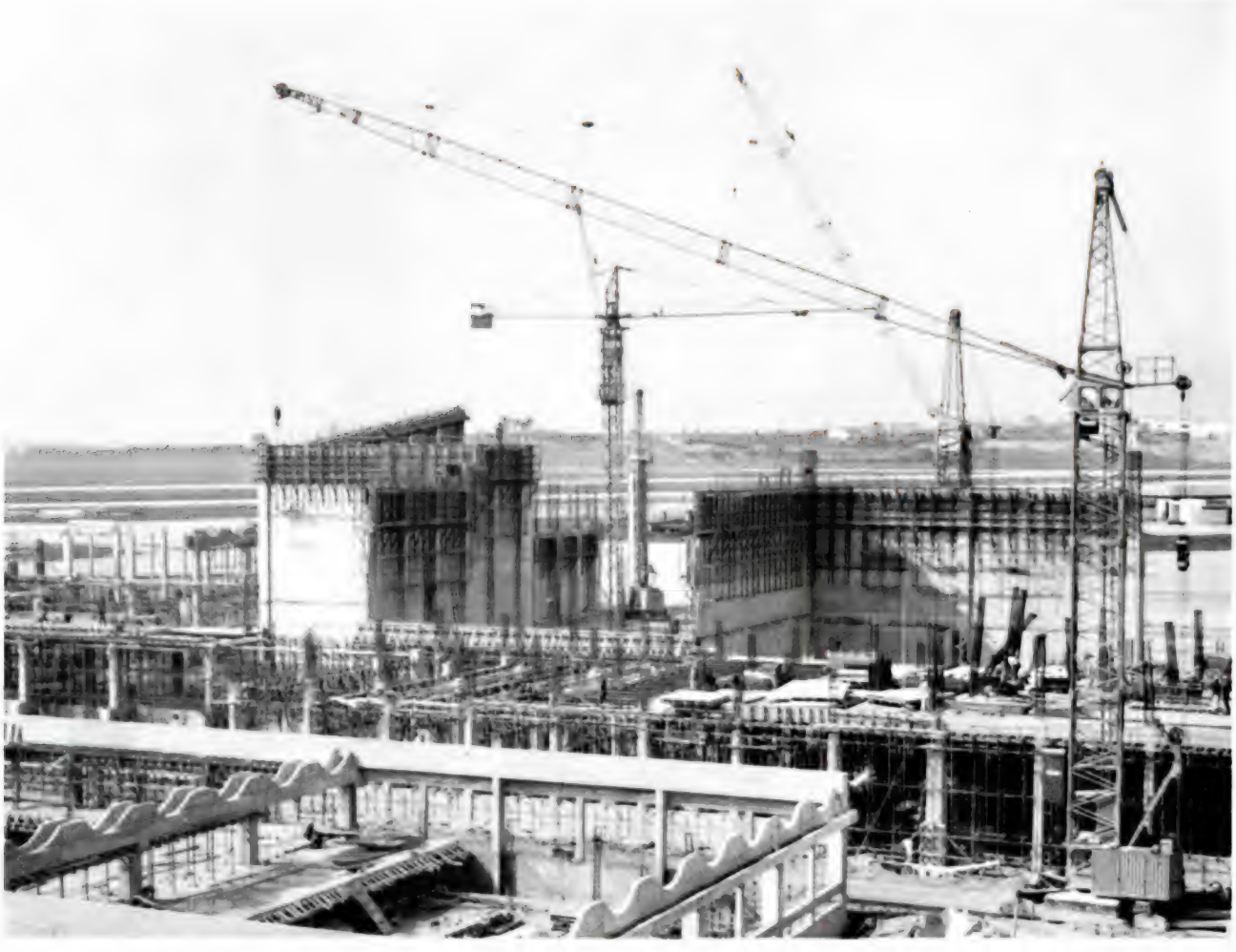

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loped with this in mind. It is also worthy of note that the numerical solution procedure discussed in the Extension Course referred to on page 17, is suitable for solving parabolic differential equations such as equation (13). Thus, this numerical procedure, which has already proved very satisfactory for zero equation turbulence models, is also applicable to one and to higher order turbulence models, provided the equations remain parabolic.

## 4. TWO-EQUATION TURBULENCE MODELS

In one-equation turbulence models, it is necessary to specify $l_D$ (and $l_\alpha$) empirically. As was the case for the mixing length, this and universality are incompatible. The logical step forward is to provide a differential equation with $l_D$ as dependent variable and determine if this raises any new problems.

Such an equation could be obtained in the same manner as L. S. G. Kovasznay and V. N. Nee obtained their equation for $\nu_t$; that is, to write down an equation of the form:

$$U \frac{\partial l}{\partial x} + V \frac{\partial l}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_l}\frac{\partial l}{\partial y}\right) + G - D \quad (16)$$

where the form of the generation and dissipation terms, G and D, are guessed. A more satisfactory procedure is to attempt to obtain an equation from first principles. It may be remembered that, in the previous section, it was assumed that dissipation of turbulent energy could be replaced by:

$$\frac{C_D\,(k)^{3/2}}{l_D}.$$

This implies that if an equation for dissipation can be derived, it can be related to $l_D$ and k provided local isotropy of the dissipating eddies is assumed.

An equation for dissipation was derived by F. H. Harlow and P. I. Nakayama (19) based on earlier derivations by P. Y. Chou (20). The derivation can be set down as follows:

The equation:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} - \overline{u_j \frac{\partial u_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \quad (17)$$

may be obtained by subtracting the time mean components from the general Navier-Stokes equations. Together with continuity it implies:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_j}\overline{(u_i u_j)} \quad (18)$$

If equation (18) is differentiated with respect to $x_l$ and multiplied by $\frac{\partial u_i}{\partial x_l}$ and thenaver aged the result is:

$$\underbrace{\frac{\partial^2 U_i}{\partial x_l \partial x_j}\overline{u_j \frac{\partial u_i}{\partial x_l}}}_{(a)} + \underbrace{\frac{\partial U_i}{\partial x_j}\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}}}_{(b)} +$$

$$+ \underbrace{\frac{\partial U_i}{\partial x_l}\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}}}_{(c)} + \underbrace{U_j \frac{\partial}{\partial x_j}\overline{\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_l}\right)^2\right]}}_{(d)} +$$

$$+ \underbrace{\overline{\frac{\partial u_j}{\partial x_l}\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}}}_{(e)} + \underbrace{\overline{u_j \frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_l}\right)^2\right]}}_{(f)} =$$

$$= \underbrace{-\frac{1}{\rho}\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial^2 p}{\partial x_l \partial x_j}}}_{(g)} + \underbrace{\nu\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial u_i}{\partial x_l}\right)}}_{(h)} =$$

$$= -\frac{1}{\rho}\overline{\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_l}\frac{\partial p}{\partial x_l}\right)} +$$

$$+ \nu \frac{\partial^2}{\partial x_j^2}\overline{\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_l}\right)^2\right]} - \nu\overline{\left[\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right]} \quad (19)$$

The two-dimensional, boundary-layer form of equation (19) is

$$\underbrace{\left(U \frac{\partial S}{\partial x} + \frac{\partial S}{\partial y}\right)}_{(d)} + \underbrace{\frac{\partial^2 U}{\partial y^2}\overline{\left(v \frac{\partial u}{\partial y}\right)}}_{(a)} +$$

$$+ \frac{\partial U}{\partial y} \overline{\left( \frac{\partial u_i}{\partial x_l} \frac{\partial u_i}{\partial x_l} \right)} + \frac{\partial U}{\partial y} \overline{\left( \frac{\partial u_i}{\partial x_l} \frac{\partial u_i}{\partial x_j} \right)} +$$

(b) (c)

$$+ \overline{\frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l} \frac{\partial u_i}{\partial x_j}} + \nu \overline{\frac{\partial S}{\partial y}} + \frac{1}{\rho} \overline{\frac{\partial u_i}{\partial x_l} \frac{\partial^2 p}{\partial x_l \partial x_i}} -$$

(e) (f) (g)

$$- \nu \frac{\partial^2 S}{\partial y^2} + \nu \left[ \overline{\frac{\partial^2 u_i}{\partial x_j \partial x_l}} \right]^2 = 0 \qquad (20)$$

with $S = \frac{1}{2} \left( \overline{\frac{\partial u_i}{\partial x_l}} \right)^2$

This equation has dissipation as dependent variable but is intractable in its present form. (It should be noted that, for simplicity, density has been assumed as unity). To make it tractable the following assumptions can be made (see refs. 19, 20 and 21):

— term (a) may be neglected from order of magnitude considerations

— terms (b) and (c), from a loose kinetic energy analogy, may be postulated in the form

$$c_1 f_1 \frac{S}{k} \overline{uv} \frac{\partial U}{\partial y}$$

— terms (e), (f) and (g) are diffusion terms and, as usual, may be approximated by a gradient term,

$$\text{i.e.} \quad \frac{\mu_t}{\sigma_S} \frac{\partial S}{\partial y}$$

— the last term in equation (20) may be approximated by

$$c_2 f_2 \frac{S^2}{k}$$

With these assumptions the equation may be written, for high Reynolds numbers, as:

$$U \frac{\partial S}{\partial x} + V \frac{\partial S}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\mu_t}{\sigma_S} \frac{\partial S}{\partial y} \right) +$$

$$+ C_1 \frac{S}{k} \mu_t \left( \frac{\partial U}{\partial y} \right)^2 + C_2 \frac{S^2}{k} = 0 \qquad (21)$$

Equation (21) has the same form as equation (17) and may be solved together with continuity, mean momentum and turbulent kinetic energy if the eddy viscosity is postulated in the form:

$$\mu_t = C_\mu \frac{(\rho k)^2}{S} \qquad (22)$$

This model requires the specification of $\sigma_k$, $\sigma_S$, $C_1$, $C_2$ and $C_\mu$, i. e. one more than in the one-equation model of the previous section; the need to specify the length scale has, however, been eliminated. This last point is important because whereas Prandtl numbers may reasonably be expected to be constant, length scales cannot.

It is the writer's opinion that this model represents the points of diminishing return of boundary layer flows with the exception of those flows where strong asymmetry is present; and even in such asymmetric cases, it may well not be worth solving additional equations. The model has been tested at Imperial College by W. P. Jones and B. E. Launder (22) and appears to be satisfactory over a wide range of flows.

It must be recognised that having derived, with some assumptions, an equation for S or $l_D$, alternative equations can be derived. For example, the equation for S may be regarded as an equation for $\frac{k^{3/2}}{l_D}$. Equations can also be derived for $k^m l_D^n$ where m and n can take on any value. D. B. Spalding and K. H. Ng (23) have preferred an equation for $(k/l_D)$ and have used it to predict the data assembled for the Stanford Conference and also for supersonic boundary layers; in both cases the results are excellent.

Also within the framework of the two-dimensional models, D. B. Spalding (24) has employed an equation for fluctuating vorticity in place of

the dissipation equation; that is, the dependent variable is $w = (\partial u/\partial y - \partial v/\partial x)^2$. They employ the equation in the form:

$$U \frac{\partial w}{\partial x} + V \frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_w} \frac{\partial w}{\partial y}\right) + $$

$$+ C_1 \frac{k}{w^{\frac{1}{2}}} \left(\frac{\partial^2 U}{\partial y^2}\right)^2 + C_2 w^{3/2} +$$

$$+ C_3 w^{\frac{1}{2}} \left(\frac{\partial U}{\partial y}\right)^2 = 0; \qquad (23)$$

the k-equation in the form,

$$\frac{U}{2} \frac{\partial k}{\partial y} + \frac{V}{2} \frac{\partial k}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_{k,t}} \frac{\partial k/2}{\partial y}\right) + \qquad (24)$$

$$+ \frac{k}{w^{\frac{1}{2}}} \left(\frac{\partial U}{\partial y}\right)^2 + C_D k w^{\frac{1}{2}} = 0,$$

and the Prandtl/Kolmogorov hypothesis in the form:

$$\mu_t = C_\mu \rho k w^{-\frac{1}{2}} \qquad (25)$$

Of the models discussed in this section, the $k-S$ and $k-k_l$ models have been subjected to the most thorough testing. Both have proved highly successful, although it remains to apply them to some important flow situations such as those with strong density gradient. These models do, however, seem very promising and it is likely that design procedures of the future will utilise two-equation models.

## 5. THREE-EQUATION TURBULENCE MODELS

Two reasons exist for considering the addition of a third turbulence equation. It will inevitably allow greater generality since it will involve more constants (or functions) which can be assigned values based on experimental information. The second reason is that, in strongly asymmetric turbulent flows, it is known that the surfaces of maximum velocity and zero shear stress will not coincide. Sometimes the distance between these surfaces is negligibly small compared with the width of the boundary layer but on other occasions the distance is not small and must be considered if accurate predictions are to be achieved. An example of a flow situation where the non-coexistance of zero shear and maximum velocity is important is flow in an annulus where one wall is smooth and the other rough. In such cases, it is necessary to add an equation for the shear stress $\overline{uv}$. An appropriate set of equations is:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{dP}{dx} + \frac{\partial}{\partial y} \overline{uv} = 0$$

$$\frac{U}{2} \frac{\partial k}{\partial x} + \frac{V}{2} \frac{\partial k}{\partial y} + \overline{uv} \frac{\partial U}{\partial y} +$$

$$+ \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_{k,t}} \frac{\partial k/2}{\partial y}\right) + (2S) = 0 \qquad (26)$$

$$U \frac{\partial S}{\partial x} + V \frac{\partial S}{\partial y} + C_1 \frac{S}{k} \overline{uv} \frac{\partial U}{\partial y} +$$

$$+ \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_S} \frac{\partial S}{\partial y}\right) C_2 \frac{S^2}{k} = 0$$

$$U \frac{\partial \overline{uv}}{\partial x} + V \frac{\partial \overline{uv}}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\mu_t}{\sigma_\tau} \frac{\partial \overline{uv}}{\partial y}\right) +$$

$$+ C_2 C_\mu k \frac{\partial U}{\partial y} + C_3 \frac{S \overline{uv}}{k} = 0$$

This set of equations requires a specification of $\sigma_{k,t}$, $\sigma_S$, $\sigma_\tau$, $C_1$, $C_2$, $C_3$ and $C_\mu$. It does not require the specification of the length scale and does not use the Prandtl-Kolmogorov viscosity law. It is useful to note that when convection and diffusion of shear stress are mad equal, equation (26) implies that:

$$C_\mu \frac{k^2}{S} \frac{\partial U}{\partial y} = \overline{uv} \quad ,$$

i.e. the Prandtl-Kolmogorov eddy viscosity law.

The above three-equation model has been tested with some succes by K. Hanjalic and B. E. Launder (25) using the computational procedure which was described in the previous course. Wall-jet data, two-dimensional plane rough-channel data and rough-annulus data have all been well predicted.

## 6. MAOY-EQUATION TURBULENCE MODELS

It should now be clear that a system of equations can be built up which contains a very large number of equations. The question must also be asked, however, is it worth doing? For boundary-layer flows, there is likely to be little merit in a system of more than three equation for the turbulent flow; and, for the vast majority of cases, two equations should suffice. Larger numbers of equations have been proposed and it useful to point out how these are utilised and to mention some points in their favour. It is likely that a greater number than three turbulence equations will be required in recirculating flows, where the more complex elliptic, time-average Navier- Stokes equations must be solved for the mean flow and the description of the included in the various equations. Thus, in this brief section, the comments are of some relevance to the separated flow situations.

In equation (13), the production of turbulent energy by the mean fllow working against normal stresses must be neglected in a two-or three-equation model. This is because there is no specific information regarding the three fluctuating velocity components. This is justified on the grounds that, in the vast majority of boundary-layer flow situations, the normal-stress production is smal compared to the shear-stress production. However, in a region close to separation, this is not true. In such cases, it may be desirable to replace the turbulence kinetic energy equation by three equtions, one each for the three components of fluctuating velocity; length scales can be formulated for each of these three equations; shear stress equations with further length scale equations can also be derived. Of course, the number of constants increases at a rate of approximately twice the number of equations if the high Reynolds number assumption is made and may be even larger if this assumption is not made. Thus, the optimisation of the constants will present formidable problems.

## 7. CONCLUSIONS

It is useful to review here the conclusions which have been drawn in the text. They may be stated as follows:

1. Zero-equation turbulence models are currently in use for design purposes. They offer greater generality than correlations or integral procedures and have been subjected to considerable testing and evaluation.

2. Two-equation turbulence models apper to offer the best compromise of generality and simplicity but have not yet been tested over a wide enough range of flow situations to allow them to be recommended for design purposes. It is likely that, in the future, such design procedures will be used.

3. In strongly asymmetric turbulent flows, three--equation turbulence models may be necessary.

4. It will be necessary in all or any of the above turbulence models to embody them in computer programs which are economical of both storage and computing time.

## NOMENCLATURE

| | |
|---|---|
| C | Constants |
| $l$ | Length sclale |
| k | Turbulent kinetic energy $\left[=\frac{\overline{u^2+v^2+w^2}}{2}\right]$ |
| p | Fluctuating pressure |
| P | Mean pressure |
| S | Dissipation of turbulent energy $\left[=\overline{\left(\frac{\partial u_i}{\partial x_i}\right)^2}\right]$ |
| T | Temperature |
| u | Fluctuating velocity in x-direction |
| U | Time mean velocity in x-direction |
| v | Fluctuating velocity in y-direction |
| V | Time mean velocity in y-direction |
| w | Time mean velocity in z-direction [also $= \overline{(\partial u/\partial y - \partial v/\partial x)^2}$] |
| x, y, z | Orthogonal coordinates: x in stream-wise direction |
| $\delta_1$ | Displacement thickness |
| $\rho$ | Density |

$\mu$ Dynamic viscosity
$\nu$ Kinematic viscosity
$\sigma$ Prandtl or Schmidt number
t Fluctuating temperature
f Frequency

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# FEIRAS, CONFERÊNCIAS, EXPOSIÇÕES E CURSOS NO MUNDO

| OBJECTIVO E PROGRAMA | LOCAL, DATAS, ORGANIZAÇÃO, INFORMAÇÕES |
|---|---|
| Réunion Commune du Comité d'Experts du RID et du Groupe d'Experts des transports de mar chandises dangereuses | Berne, Maio-71<br>E C E |
| Formação Básica de Encarregados na Indústria Actualização em Técnicas Matemáticas | Lisboa, 24-28 Maio-71<br>COPRAI |
| Simpósio Sobre Identificação e Medição de Poluidores do Meio Ambiente | Ottawa, Junho-71<br>Mr. M. K. Ward, Int. Symposium NRC, Ottawa 7, Ontário, Canadá |
| Problemas de Produção<br>Programa :<br>Planeamento e Programação, Conservação e Substituição do Equipamento | Lisboa, Junho-71<br>INII |
| Problemas Actuais da Indústria da Madeira (Jornadas)<br>Tema :<br>Carpintaria e Parquetaria | Aveiro, Junho-71<br>INII |
| Comunicações e Relações no Trabalho | Luanda, Junho-71<br>INII e AIA<br>Lourenço Marques, Junho-71<br>INII e AIM |
| Funções e Tarefas dos Chefes de Serviços | L. Marques, Junho-71<br>INII e AIM |
| Hauptversammlung der Deutschen Gesellschaft Für Metallkundf<br>Tema :<br>Metalurgia | Lausana, Suiça, 1-4 Junho-71<br>Deutsche Gesellschaft Für Metallkund e. V., An Der Alteburger Mühle 12, 5-Köln51, Alemanha. |
| Simpósio sobre as alterações indesejáveis das substâncias gordas | Gdansk, Polónia, 3-5 Junho-71<br>Dep. de Química e Tecnologia de Corpos Gordos, 11 ul. Majakowskiego (Politechnika) |
| Osterreichische Eisenhüttentag 1971<br>Tema :<br>Ferro e Aço | Loeben, Austria, 3-5 Junho-71<br>Montanistische Hochschule, A-8700 Loeben (Steiermark) Austria |
| Congrès International I. C. F.<br>Tema :<br>Ciência e Técnica no Ano 2000 | Paris, 6-13 Junho-71<br>Societé des Ingénieurs Civils de France (Service ou Congres) 19, Rue Blanche, 75-Paris-9.e, França |
| Present and Future of the Electric Arc Furnace in Steelmaking. | Cannes, 7-9 Junho-71<br>Institut de Recherches de la Sidérurgie, 57 Maizières-Les-Metz, França |
| Annual Assembly of the International Institute of Welding | Estocolmo, 7-12 Junho-71<br>International Institute of Welding, 54 Princess Gate, London, S.W. 7, Inglaterra |

# HERÓIS DA LUBRIFICAÇÃO

## Ben-Hur

Toda a gente sabe quem foi Ben-Hur, o quadriguista mais famoso do seu tempo.

O que poucos sabem é que Ben-Hur era um dos mais antigos chefes de manutenção da Companhia Grega de Carros de Corrida. E foi por ele ser homem dotado de muita experiência que conseguiu «limpar» o Grand Prix Anual de Quadrigas, no Estádio de Antióquia.

Messala, o grande favorito, representava as cores romanas. As apostas eram de 8 para 1.

Ben não passava de um desconhecido e ninguém se atrevia a investir o seu rico dinheirinho numa incógnita

## Ben-Hur

daquelas. Mas Ben-Hur teve uma súbita inspiração na manhã do próprio dia da corrida.

Embora sabendo que os seus cavalos não corriam tanto como os de Messala, que eram árabes e tinham a culassa rebaixada, ocorreu-lhe que uma lubrificação eficaz das rodas das quadrigas talvez fosse uma vantagem.

O problema pedia resolução urgente, mas... onde é que lhe seria possível obter uma assistência técnica rápida?

Consultou a lista de fornecedores de lubrificantes, decidiu-se Ben, depois de um meticuloso exame, a pedir uma mãozinha à MOBIL. E porquê? Porque a MOBIL tinha um corpo muito completo de pessoal habilitado. Tudo levava, pois, a crer que lhe enviariam um Delegado a tempo.

Ben-Hur tinha razão de assim pensar.

O Delegado da MOBIL veio a correr, o caminho todo, com um frasco de «Mobilinicum XVII — Massa Lubrificante Especial para Eixos de Carros de Corrida».

A corrida começou e o resultado foi o previsto (o previsto por Ben-Hur, que a multidão, essa, apostara em Messala...). Foi a lubrificação dos eixos do carro de Ben que venceu, incontestàvelmente, a prova. Messala perdeu por seis carros e, desgostoso, acabou os seus dias a conduzir carrinhos de mão...

Moral para Messala: em Roma sê romano, em corridas de carros sê MOBIL!

Moral para os responsáveis pela manutenção nos nossos dias: não é só o custo que identifica um lubrificante. A assistência que funciona por detrás dele é decisiva — e de que maneira!

A MOBIL, excepcionalmente bem provida de Delegados competentes e tècnicamente muito bem orientados, pode garantir aos clientes a assistência de que eles necessitam.

Messala tinha toda a assistência do seu lado, mas essa não era «assistência» MOBIL...

Os Delegados da MOBIL estão ao vosso dispor — com os melhores lubrificantes que podeis comprar e com a competência técnica de que precisais.

Eles estão sempre onde são precisos!

Mobil

C. D. U. 537.226

# CONCEPÇÃO FÍSICA E INTERPRETAÇÃO ANALÍTICA DAS PERDAS DIELÉCTRICAS

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RESUMO

*Estudam-se as perdas por condução, devidas à polarização e causadas por descargas parciais. Para isso analisam-se as variações da condutividade dieléctrica e da permitividade e calcula-se o esquema equivalente de um dieléctrico com cavidades.*

SYNOPSIS

*The losses of conduction, of polarization and by partial discharges are studied. For that we analyse the variations of the conductivity and the permitivity, and an equivalent diagram of the partial discharges in voids is calculated.*

## 1. INTRODUÇÃO

Em qualquer sistema eléctrico, por mais simples que seja, surgem problemas de isolamento. O conhecimento dos materiais isolantes aplicáveis a fim de se conseguir o conveniente funcionamento é uma preocupação do engenheiro que projecta e que executa.

No que se refere aos materiais dieléctricos é essencial a compreensão das perdas. Existem hoje teorias que dão em boa parte resposta a essas inquietantes interrogações, permitindo-se uma representação física racional, enquadrada nas modernas concepções da constituição da matéria, e uma tradução analítica fundamental na descrição dos conceitos físicos respectivos.

Um trabalho de sistematização é em última análise o objectivo do presente artigo. Para isso estabelecem-se as expressões clássicas para o dieléctrico ideal, introduzindo-se a noção de permitividade (1) complexa para atender às perdas de polarização [1], e descrevem-se as ideias basilares da teoria de Debye que deu elevado impulso ao estudo dos dieléctricos [2]. Finalmente elabora-se um modelo aproximado para se deduzirem as perdas de ionização [3].

(1) Usa-se o termo permitividade para denominar a grandeza $\varepsilon$ normalmente designada constante dieléctrica por não ser de facto um valor «constante» (depende da frequência e da temperatura em geral).

## 2. CONDENSADOR ELÉCTRICO DE VÁCUO

Seja um condensador cujo dieléctrico é o vácuo, portanto constituído sem matéria. Quando a tensão aplicada aos eléctrodos do condensador for v a carga eléctrica acumulada na superfície dos eléctrodos é $Q = C_0 v$, onde $C_0$ é a capacidade electroestática do condensador, sendo a intensidade de corrente no circuito exterior $i = dQ/dt$ ou $i = C_0 . dv/dt$.

Se a tensão for alternada sinusoidal $v = \hat{V} \sin \omega t$, onde $\hat{V}$ é o seu valor de pico, então $i = \hat{I} \cos \omega t$ com $\hat{I} = \omega C_0 \hat{V}$ ou, em simbologia complexa, $\dot{I} = j\omega C_0 \dot{V}$, apresentando-se uma desfasagem da corrente de $\pi/2$ rad em avanço relativamente à tensão e portanto a potência activa, definida pelo valor médio do produto v. i, é nula dado que as grandezas v e i são ambas sinusoidais puras.

Admitindo um condensador plano a *capacidade geométrica* $C_0$, assim chamada por se caracterizar apenas pelas dimensões do condensador, será $C_0 = \varepsilon_0 . A/d$ onde $\varepsilon_0$ é uma constante universal $\varepsilon_0 = 8{,}85\ pFm^{-1}$, A é a área da face dos eléctrodos e d a distância entre eles.

Define-se *densidade de fluxo eléctrico* ou *deslocamento eléctrico* pela carga eléctrica por unidade de superfície $D = q$, onde $q = Q / A$ se for Q a carga total de cada placa do condensador (em rigor, pela lei de Gauss, $Q = \oint_S \vec{D} . \vec{dS}$ onde S é qual-

quer superfície fechada que envolve a carga Q), sendo vectorialmente $\vec{D}_0 = \varepsilon_0 \vec{E}_0$, isto é, o deslocamento eléctrico está em fase com o campo eléctrico.

Sob o ponto de vista prático o ar comporta-se com aproximação suficiente, se não ocorrerem fenómenos de disrupção completa ou incompleta, como se se tratasse de vácuo: para o ar à pressão atmosférica a sua permitividade é 0,6 ‰ maior que o valor $\varepsilon_0$ do vazio.

## 3. DIELÉCTRICO PERFEITO

Introduza-se no condensador um dielétrico de permitividade relativa $\varepsilon$ e de condutividade eléctrica nula (dieléctrico perfeito). Fìsicamente a introdução deste material em substituição do vácuo provoca uma redução do campo eléctrico de $\varepsilon$, ou seja $\vec{E} = \vec{E}_0/\varepsilon$, sendo $\vec{D} = \vec{D}_0$. Este facto resulta da *polarização* do dieléctrico sob o efeito do campo aplicado exterior que, orientando os dipolos elementares da sua estrutura na direcção do campo, provoca um campo de polarização $\vec{E}_P$, em oposição ao campo aplicado, tal que $\vec{E} = \vec{E}_0 - \vec{E}_P = \vec{E}/\varepsilon$.

Mas para a mesma tensão aplicada mantém-se o campo eléctrico $(\vec{E} = \vec{E}_0)$ pelo que deverá aumentar de $\varepsilon$ a densidade do fluxo eléctrico, $\vec{D} = \varepsilon \vec{D}_0$ donde $\vec{D} = \varepsilon_0 \varepsilon \vec{E}$. O deslocamento eléctrico corresponde igualmente a uma densidade de fluxo eléctrico no vazio sobreposta a uma densidade de fluxo de polarização, chamada *vector de polarização* $\vec{P} = \vec{D} - \vec{D}_0$, donde $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ ou $\vec{D} = \varepsilon_0 \varepsilon \vec{E}$ com $\vec{P} = \varepsilon_0 (\varepsilon - 1) . \vec{E}$, onde $\chi = \varepsilon - 1$ se chama *susceptibilidade dieléctrica* de modo que $\vec{P} = E_0 \chi \vec{E}$.

Daqui resulta que para se manter igual a intensidade do campo eléctrico com o vazio ou um dado material de permitividade $\varepsilon$ exige-se uma carga eléctrica $\varepsilon$ vezes maior no caso de existir matéria. Os dipolos de facto constituem longas cadeias de partículas, alinhadas devido à polarização, terminando com uma carga eléctrica positiva junto do eléctrodo negativo e uma carga eléctrica negativa na extremidade próxima do eléctrodo positivo. Estas cargas eléctricas superficiais do dieléctrico, opostas às correspondentes cargas eléctricas dos eléctrodos, neutralizam parte da carga eléctrica das placas. Se for Q a carga eléctrica total apenas uma parte $Q/\varepsilon$ contribui para neutralizar a tensão aplicada sendo a restante carga eléctrica $Q\ (1 - 1/\varepsilon)$ neutralizada pela polarização.

Pela definição de deslocamento eléctrico é $D = q$ e por conseguinte $q\ dA = \vec{D} . d\vec{S}$, onde S é qualquer superfície que envolve A (lei de Gauss). A carga eléctrica relativa ao campo aplicado $q/\varepsilon$ no caso do vácuo conduz a $(q/\varepsilon) . dA = \varepsilon_0 \vec{E} . d\vec{S}$. A restante carga eléctrica define anàlogamente o vector de polarização $\vec{P}$ tal que $q\ (1 - 1/\varepsilon)\ dA = \vec{P} . d\vec{S}$ pelo que $q\ dA = \varepsilon_0 \vec{E} . d\vec{S} + \vec{P} . d\vec{S}$ ou seja $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$.

O fenómeno da polarização equivale a um aumento da capacidade do valor $\varepsilon$, isto é, $C = \varepsilon C_0$, continuando a verificar-se as mesmas relações para a carga eléctrica e corrente em função da tensão tendo em atenção o novo valor da permitividade, $Q = C v$, $i = C . dv/dt$, $\dot{I} = j \omega C \dot{V}$.

Devido contudo ao fenómeno de polarização, mesmo num dieléctrico perfeito (de condutividade eléctrica nula), existem as chamadas *perdas dieléctricas de polarização* que se traduzem numa componente óhmica da corrente. Este facto pode estudar-se por intermédio da *permitividade complexa* $\varepsilon = \varepsilon' - j \varepsilon''$, isto é, o deslocamento eléctrico não se encontra em fase com o campo eléctrico mas sim em atraso devido à inércia da polarização, que determina uma *capacidade complexa* $\dot{C} = (\varepsilon' - j \varepsilon'') . C_0$, sendo nestas condições $\dot{I} = j \omega (\varepsilon' - j \varepsilon'') C_0 \dot{V}$ e portanto com $\dot{I} = \dot{I}_R + j\dot{I}_C$ deduz-se $\dot{I}_R = \omega \varepsilon'' C_0 \dot{V}$ e $\dot{I}_C = \omega \varepsilon' C_0 \dot{V}$ donde o *factor de perdas de polarização* $\operatorname{tg} \delta_P = \dot{I}_C / \dot{I}_R$ resulta $\operatorname{tg} \delta_P = \varepsilon''/\varepsilon'$.

A presença da corrente activa corresponde de facto a uma *condutividade dieléctrica de polarização* $\sigma_P$ que se pode deduzir da lei d'Ohm $\dot{I}_R = G_P \dot{V}$, sendo $G_P$ a correspondente condutância $G_P = \sigma_P A/d$ e juntamente com $C_0 = \varepsilon_0 A/d$ se permite concluir $\sigma_P = \omega \varepsilon_0 \varepsilon''$ e $\operatorname{tg} \delta_P = \sigma_P / \omega \varepsilon_0 \varepsilon'$.

Note-se porém que $\varepsilon''$ apenas tem valores apreciáveis, e portanto as perdas de polarização, quando a inércia da polarização for significativa, o que fìsicamente equivale a um não acompanha-

mento do ritmo do campo aplicado, necessàriamente para elevadas frequências. Este fenómeno será objecto de análise detalhada.

## 4. DIELÉCTRICO REAL

Os materiais dieléctricos são porém mais ou menos imperfeitos, possuindo uma condutividade eléctrica não nula, embora muito reduzida, que provoca as denominadas *perdas dieléctricas de condução iónica.*

Clàssicamente o problema das perdas dieléctricas é encarado através do aparecimento de uma componente activa da corrente total pelo que, para $v = \hat{V} \sin \omega t$, será $i = \hat{I} \sin (\omega t + \varphi)$ onde $\varphi < \pi/2$ rad e $\dot{I} = \dot{I}_R + j\,\dot{I}_C$ com $\operatorname{tg} \varphi = \dot{I}_C/\dot{I}_R$. Assim a potência activa, relativa às perdas dieléctricas, deixa de ser nula, $P = (1/T) \cdot \int_0^T v\,i\,dt = \hat{V}\,\hat{I} \cos \varphi/2$ ou em valores eficazes, $P = VI \cos \varphi$

Em geral caracterizam-se as perdas dieléctricas através do *ângulo de perdas* $\delta = \pi/2 - \varphi$, donde $P = VI \sin \delta$, com $I \cos \varphi = I_R = I_C \operatorname{tg} \delta$ e $I_C = \omega CV$, concluindo-se a expressão clássica

$$P = \omega C V^2 \operatorname{tg} \delta \qquad (1)$$

que traduz a proporcionalidade das perdas dieléctricas com a frequência $\omega$, a capacidade C do dieléctrico (ou a sua permitividade $\varepsilon$), o quadrado da tensão aplicada e o factor de perdas $\operatorname{tg} \delta$.

Este resultado é importante em alta tensão pois sendo V muito elevada, e para uma dada frequência (50 Hz no caso das redes de distribuição de energia eléctrica) a redução das perdas dieléctricas só se pode conseguir pelas características do material dieléctrico: por um lado pequena permitividade $\varepsilon$ e por outro lado baixa condutividade (ou seja reduzido ângulo de perdas $\delta$). Trata-se pois de um problema fundamental da tecnologia dos materiais isolantes.

Note-se a aproximação $P \simeq U I \operatorname{tg} \delta$ no caso de grandezas sinusoidais, válida para pequenos valores de $\delta$ pois então $\operatorname{tg} \delta \simeq \sin \delta = \cos \varphi$ ou seja $I \simeq I_C$.

O *esquema eléctrico equivalente* do condensador pode conceber-se pelo paralelo de uma resistência R de perdas de condução com a capacidade C, normalmente utilizado, mas nalguns casos interessa considerar o esquema série da resistência $R_s$ com C, havendo a relação $R R_s = 1/(\omega C)^2$ entre R e $R_s$ admitindo a igualdade das capacidades de ambos os esquemas como fàcilmente se calcula pela equivalência das impedâncias dos dois esquemas.

Atendendo, contudo, às perdas de polarização o esquema equivalente refere-se ao paralelo de R com a capacidade complexa $\dot{C}$ pelo que $\dot{I} = \dot{V}/R + j\omega(\varepsilon' - j\varepsilon'')\,C_0\,\dot{V}$ sendo a componente óhmica da corrente global $\dot{I}_R = (1/R + \omega\varepsilon'' C_0)\,\dot{V}$ e a componente reactiva $\dot{I}_C = \omega\varepsilon' C_0 \dot{V}$, donde

$$\operatorname{tg} \delta = \frac{1}{\omega \varepsilon' R C_0} + \frac{\varepsilon''}{\varepsilon'} \qquad (2)$$

que se pode considerar $\operatorname{tg} \delta = \operatorname{tg} \delta_C + \operatorname{tg} \delta_P$, a primeira parcela relativa às perdas de condução iónica $\operatorname{tg} \delta_C = 1/\omega \varepsilon' R C_0$ e a segunda parcela correspondente às perdas de polarização $\operatorname{tg} \delta_P = \varepsilon''/\varepsilon'$.

## 5. CONDUTIVIDADE DIELÉCTRICA

Após o instante de aplicação de uma tensão contínua a um dieléctrico constata-se em *função do tempo* uma variação da condutividade em quatro zonas (fig. 1-a), a saber:

Zona I — Em menos de $10^{-2}$ seg após o instante de aplicação da tensão ao dieléctrico a condutividade decresce até se estabilizar devido à orientação dos dipolos da estrutura do material, definindo-se a *condutividade de polarização.*

Zona II — Entre $10^{-2}$ e 10 seg os electrões livres carregam os eléctrodos do condensador, correspondendo de facto à *condutividade a 50 Hz.*

Zona III — Para tempos de 10 a $10^3$ seg dá-se uma forte redução da condutividade devida a duas causas: empobrecimento de rápidas partículas livres transportadoras de carga e grande concentração de cargas eléctricas nos eléctrodos. Poder-se-á referenciar este intervalo por *condutividade dieléctrica transitória.*

Zona IV — Quando o tempo de aplicação da tensão contínua excede $10^3$ seg atinge-se um valor constante que determina a chamada *condutividade estacionária.*

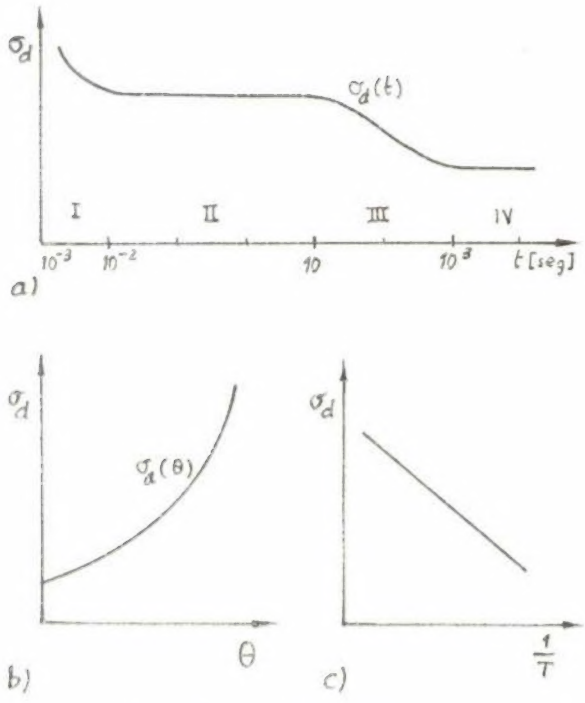


Fig. 1 — Condutividade dieléctrica.
*a*) Em função do tempo.
*b*) Em função da temperatura (escala linear).
*c*) Em função do inverso da temperatura absoluta (escala semi-logarítmica).

A dependência da temperatura da condutividade dieléctrica em corrente contínua e em corrente alternada à frequência industrial é regida pela *lei de Van't Hoff*

$$\sigma_d = \sigma_{do} \exp\left(-\frac{W}{KT}\right) \qquad (3)$$

onde $\sigma_{do}$ é uma constante dependente da concentração iónica, W a energia térmica de activação, K a constante universal de Boltzmann e T a temperatura absoluta, cuja representação gráfica (fig. 1-b, c) em escala semi-logarítmica em função de 1/T é uma recta pois $\ln \sigma_d = \ln \sigma_{do} - W/KT$.

## 6. POLARIZABILIDADE

De um modo genérico existem os seguintes mecanismos de polarização:

1 — *Polarização de orientação:* Nos dieléctricos sólidos e líquidos podem existir moléculas polares, isto é, cujos centros de gravidade das cargas eléctricas positivas e negativas não coincidem (exemplo: no ácido clorídrico existe numa deslocação de carga eléctrica entre os átomos H e Cl). Sob o efeito de um campo eléctrico aplicado exteriormente, estas moléculas rodam em torno dos seus eixo de simetria de modo a que cada dipolo tende a alinhar-se com o campo.

2 — *Polarização de distorsão:* Nas moléculas polares o campo eléctrico, além do efeito de rotação que define a polarização de orientação, pode deslocar os átomos entre si alterando as distâncias que os separam, e por consequência o momento dipolar da molécula, fenómeno que constitui a *polarização atómica*. Outro tipo de polarização de distorsão pode verificar-se ainda nos átomos pois sendo constituídos por uma carga eléctrica positiva central (núcleo) envolvida por uma núvem de electrões com determinadas simetrias definidas pelos seus estados quânticos, quando um campo eléctrico é aplicado os electrões são ligeiramente deslocados nas suas órbitas em relação aos núcleos induzindo um momento dipolar nos átomos que caracteriza a *polarização electrónica*, também chamada *polarização óptica* porquanto os dipolos produzidos pela deslocação elástica dos electrões têm frequências naturais iguais ou superior às da luz visível. Sob o ponto de vista técnico este tipo de polarização é de menor importância.

3 — *Polarização interfacial:* Um dieléctrico sólido não é homogéneo em toda a sua extensão, havendo zonas cristalinas, zonas amorfas, impurezas, deslocações, etc. Partículas livres transportadoras de carga eléctrica sob a acção de um campo eléctrico aplicado migram através da estrutura cristalina podendo acumular-se em defeitos estruturais que com uma carga eléctrica imagem induzida num eléctrodo originam um momento dipolar. Nos dieléctricos líquidos com comportamento newtoneano não se verifica este tipo de polarização.

O momento elementar de um dipolo constituído por duas cargas eléctricas de diferente polaridade $\pm Q$ à distância d é $m = Qd$ e tem a direcção da carga positiva para a negativa. Pode no entanto assumir-se que este momento é proporcional a um *campo eléctrico local interno* ao dieléctrico $E_{in}$ através de uma constante de proporcionalidade $\alpha$, chamada *polarizabilidade* do dipolo, sendo $\vec{m} = \alpha \vec{E}_{in}$ e tendo $\alpha$ o significado do momento dipolar médio por unidade de intensidade de campo eléctrico.

Considerando n dipolos elementares por unidade de volume com o momento elementar médio m

o vector polarização será $\vec{P} = n\vec{m}$ e daí a equação de Clausius $\vec{P} = n\alpha\vec{E}_{in}$ sendo a polarizabilidade $\alpha$ igual à soma das polarizabilidades individuais devidas a cada tipo de polarização: $\alpha = \alpha_o + \alpha_a + \alpha_e + \alpha_f$.

## 7. PERMITIVIDADE

Um dipolo eléctrico que pode rodar livremente alinha-se com um campo aplicado e se este é alternado o dipolo roda ao ritmo do campo, ocorrendo uma ressonância quando a frequência do campo aplicado for igual à frequência de rotação natural do dipolo. Se porém o dipolo está num meio viscoso, provocando um certo amortecimento por atrito, a sua resposta ao campo eléctrico é do tipo relaxação, com uma certa constante de tempo, pelo que resulta um atraso de fase da rotação relativamente ao campo aplicado.

Mas um dipolo eléctrico pode vibrar além disso pela variação da distância entre as cargas eléctricas devido ao campo aplicado, vibração encarada como um oscilador harmónico linear, dando-se também neste caso uma possível ressonância ou relaxação de acordo com o amortecimento.

Esta análise evidencia a dependência do valor da permitividade com a frequência, problema resolvido pela teoria de Debye. Além disso também depende da temperatura, esclarecendo-se igualmente pela mesma teoria o respectivo fenómeno de relaxação.

### 7.1 Influência da frequência na permitividade

a) *Devido à polarização de orientação*

Quando um dieléctrico é submetido a um campo eléctrico alternado o fenómeno da polarização apresenta uma certa inércia que origina um certo atraso do deslocamento eléctrico em relação ao campo eléctrico aplicado, o que se traduz por uma permitividade complexa: $\vec{D} = \varepsilon_o(\varepsilon' - j\varepsilon'')\vec{E}$. Para elevadas frequências a orientação dos dipolos ao ritmo do campo não se pode realizar devido à inércia, pelo que $\varepsilon''$ se anula.

Este comportamento em função da frequência exprime-se analìticamente por uma relação do tipo $\varepsilon(\omega) = \varepsilon_\infty + \int_0^\infty \alpha(t).\exp(j\omega t)\,dt$, onde $\varepsilon_\infty$ é o valor da permitividade à frequência infinita, sendo òbviamente constante, e $\alpha(t)$ o factor de queda de $\varepsilon(\omega)$ em consequência do atraso da polarização em referência ao campo, o que equivale à redução da polarização quando se retira repentinamente um campo estacionário aplicado ao dieléctrico ou à subida da polarização quando se aplica repentinamente esse campo estacionário, portanto num regime transitório caracterizado por um tempo $\tau$ chamado *tempo de relaxação*.

Debye propôs uma forma exponencial para este factor de queda da permitividade $\alpha(t) = \alpha(0).\exp(-t/\tau)$, onde o tempo de relaxação $\tau$, característico do dieléctrico, pode ser função da temperatura mas não do tempo. No caso da corrente contínua, $\omega = 0$, seja $\varepsilon(0) = \varepsilon_z$ e por outro lado $\varepsilon_z = \varepsilon_\infty + \tau.\alpha(0)$ donde $\alpha(t) = [(\varepsilon_z - \varepsilon_\infty)/\tau].\exp(-t/\tau)$.

Integrando $\varepsilon(\omega) = \varepsilon_\infty + \int_0^\infty [(\varepsilon_z - \varepsilon_\infty)/\tau].\exp[(j\omega - 1/\tau)t]\,dt$ obtém-se $\varepsilon(\omega) = \varepsilon_\infty + (\varepsilon_z - \varepsilon_\infty)/(1 - j\omega\tau)$ e, por separação das componentes real e imaginária, as chamadas *equações de Debye*

$$\varepsilon' = \varepsilon_\infty + \frac{\varepsilon_z - \varepsilon_\infty}{1 + (\omega\tau)^2} \qquad (4)$$

$$\varepsilon'' = \frac{(\varepsilon_z - \varepsilon_\infty)\,\omega\tau}{1 + (\omega\tau)^2} \quad . \qquad (5)$$

Pela representação gráfica (fig. 2-a) de $\varepsilon'(\omega)$ e $\varepsilon''(\omega)$ verifica-se a constância de $\varepsilon'$ para baixas frequências (Zona I) e para muito altas frequências (Zona III) havendo uma zona intermédia de relaxação na qual se manifesta $\varepsilon''$ (Zona II).

b) *Devido à polarização de distorsão*

A polarização de distorsão resulta da deslocação de electrões (polarização óptica) e de átomos (polarização atómica) por acção de um campo eléctrico aplicado.

Bàsicamente as cargas eléctricas elàsticamente ligadas possuem uma posição de equilíbrio que será retomada quando o campo aplicado deixar de existir por meio de oscilações amortecidas em torno da correspondente posição de equilíbrio. Este modelo permite antever uma ressonância, na

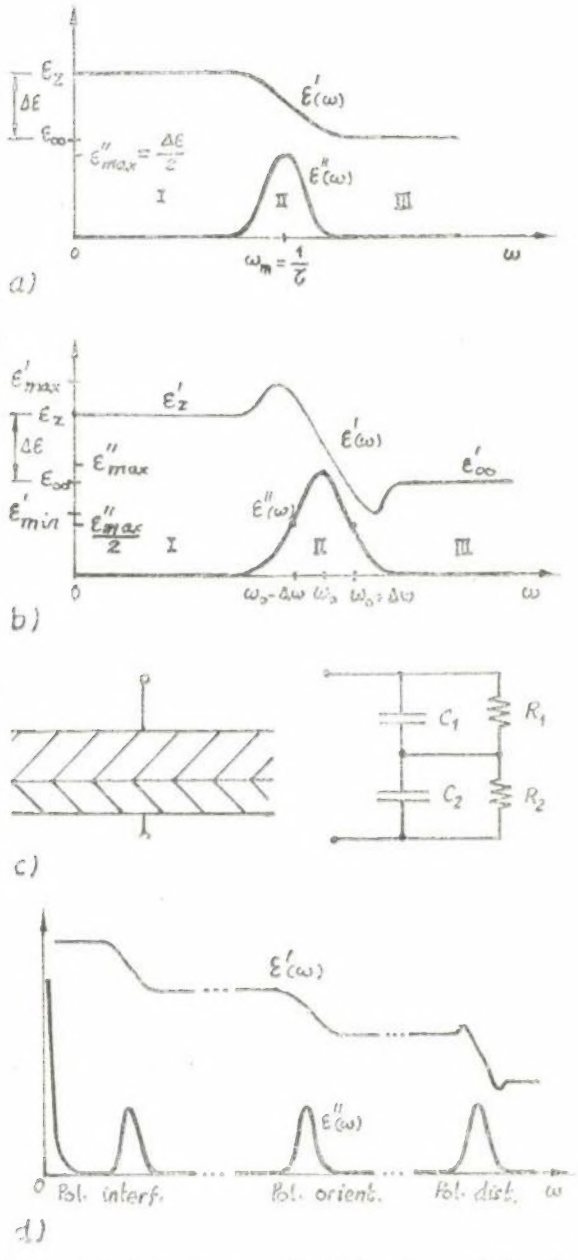


Fig. 2 — Variação da permitividade de um dieléctrico com a frequência.
*a*) Devido à polarização de orientação.
*b*) Devido à polarização de distorsão.
*c*) Esquema de um dieléctrico heterogéneo.
*d*) Variações de $\varepsilon'$ e $\varepsilon''$ com a frequência.

qual se absorve uma energia máxima, se o campo corresponde à frequência própria.

Considerando portanto o comportamento das cargas eléctricas como osciladores harmónicos lineares a equação do movimento na direcção x será $x'' + 2A\,\omega_0 x' + \omega_0^2 x = 0$, com o factor de amortecimento A e a frequência angular natural $\omega_0$, cuja solução, para pequenos amortecimentos, é do tipo $x = x_0 . \exp(-At) . \cos\omega_0 t$, definindo-se o tempo de relaxação $\tau$ do oscilador por $\tau = 1/A$ correspondente ao tempo em que a amplitude de oscilação se reduz a $1/e$ do seu valor máximo.

Froehlich propôs uma redução da polarização semelhante à de Debye afectada de $\cos(\omega_0 t + \psi)$, onde $\psi$ é o ângulo de fase relativo ao atraso da polarização em relação ao campo aplicado, que permite concluir sobre a ressonância (fig. 2-b): apresenta-se um máximo $\varepsilon'_{max} = \varepsilon_\infty + (\Delta\varepsilon/4) . \omega_0 \tau$ e um mínimo $\varepsilon'_{min} = \varepsilon_\infty - (\Delta\varepsilon/4)\,\omega_0 \tau$, com a diferença $\Delta\varepsilon$ entre os valores de $\varepsilon'$ antes e depois do período de relaxação, enquanto $\varepsilon''_{max} = (\Delta\varepsilon/2) . \omega_0 \tau$ para a frequência $\omega_0$ de modo a que $\varepsilon''_{max}/2$ corresponde às frequências $\omega_0 \pm 1/\tau$.

c) *Devido à polarização interfacial*

A heterogeneidade de um dieléctrico refere-se na sua forma mais simples (num caso mais complexo o raciocínio não se altera em princípio) a duas camadas, com uma superfície de discontinuidade, caracterizadas pelas permitividades $\varepsilon'_1$ e $\varepsilon'_2$, condutividades $\sigma_1$ e $\sigma_2$ e espessuras $d_1$ e $d_2$, respectivamente.

Quando um campo é aplicado o comportamento do dieléctrico traduz-se pela admitância equivalente $Y = Y_1 . Y_2/(Y_1 + Y_2) = j\omega(\varepsilon' - j\varepsilon'')C_0$, com $\tau_1 = R_1 C_1$, $\tau_2 = R_2 C_2$ e $\tau = (R_2\tau_1 + R_1\tau_2)/R$, sendo $R = R_1 + R_2$, que conduz às relações seguintes:

$$\varepsilon' = \varepsilon_\infty + \frac{\varepsilon_z - \varepsilon_\infty}{1 + (\omega\tau)^2} \qquad (6)$$

que é exactamente igual à expressão de $\varepsilon'$ no caso da polarização de orientação (4), quer dizer, a medida da parte real da permitividade relativa dá a soma das contribuições das polarizações interfacial e de orientação, não sendo possível separá-las;

$$\varepsilon'' = \frac{1}{\omega R C_0} + \frac{(\varepsilon_z - \varepsilon_\infty)\,\omega\tau}{1 + (\omega\tau)^2} \qquad (7)$$

que apresenta um termo adicional ao valor de $\varepsilon''$ no caso da relaxação de Debye (5), inversamente proporcional a $\omega$ (tende para infinito quando $\omega$ tende para zero numa evolução hiperbólica),

isto é, a polarização interfacial pode distinguir-se da relaxação por orientação pela observação da variação de $\varepsilon''$ abaixo da frequência de relaxação.

### 7.2 Influência da temperatura na permitividade

Num dieléctrico líquido (caso prático: óleo de transformadores) não sujeito a nenhum campo eléctrico os dipolos têm uma distribuição tal que o momento resultante é nulo. Se se aplica numa dada direcção um campo eléctrico constante então os dipolos elementares orientam-se de maneira a que o momento total deixa de ser nulo. Se se voltar a anular o campo eléctrico, em virtude do movimento térmico molecular (Browniano) o momento global torna outra vez a anular-se.

Sob as hipóteses de Debye das moléculas serem esféricas de raio $a$, de um só tipo, sem acção entre si e num fluido viscoso contínuo não polar de viscosidade $\eta$, através da lei de Stockes, que define a constante de atrito B do movimento das moléculas dipolares (sendo o momento m produzido por um campo eléctrico fazendo um ângulo $\theta$ com o eixo do dipolo de expressão $m = B\, d\theta/dt$) com o valor $B = 8\pi\eta a^3$, deduz-se para o tempo de relaxação, tendo em atenção ao movimento Browniano que afecta inevitàvelmente o ritmo de variação do ângulo com o campo, $\tau = B/2kT$, ou seja

$$\tau = \frac{4\pi\eta a^3}{kT} \qquad (8)$$

expressão que indica o resultado interessante de que a variação do tempo de relaxação $\tau$ com a temperatura é proporcional à relação do coeficiente de viscosidade e da temperatura absoluta.

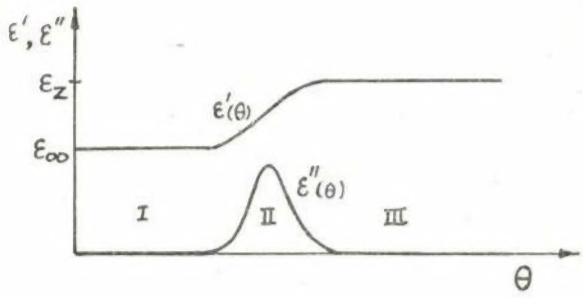


Fig. 3 — Variação da permitividade de um dieléctrico com a temperatura.

A representação gráfica (fig. 3) de $\varepsilon'$ e $\varepsilon''$ em função da temperatura apresenta três zonas distintas:

Zona I — para baixas temperaturas a viscosidade é muito elevada pelo que os dipolos estão «congelados»;

Zona II — para temperaturas intermédias a viscosidade permite a orientação dos dipolos mas o campo eléctrico $\vec{E}$ não está em fase com $\vec{D}$ pelo que surgem perdas de polarização ($W = \vec{E}\,\vec{D}/2$);

Zona III — para altas temperaturas o coeficiente de viscosidade é muito reduzido pelo que $\vec{E}$ está em fase com $\vec{D}$, sendo $\operatorname{tg}\delta_P = 0$.

## 8. PERDAS DIELÉCTRICAS

Em corrente alternada podem ocorrer três *tipos de perdas dieléctricas*: perdas de condução iónica, relativas a uma autêntica condutividade eléctrica, perdas de polarização, devidas aos diferentes aspectos da polarização do dieléctrico sob a acção do campo eléctrico aplicado, e perdas de ionização, para tensões suficientemente elevadas e causadas por descargas parciais.

### 8.1 Perdas de condução

No caso de líquidos e gases estas perdas resultam do movimento migratório permanente de electrões e iões. No caso de sólidos as partículas embora não emigrem sofrem deslocações.

Para a mobilidade b das partículas transportadoras da carga q e de concentração N a condutividade será $\sigma_d = N b q$ sendo a densidade de corrente $\vec{J} = \sigma_d \vec{E}$ ou $\vec{J} = N b q \vec{E}$ e as perdas específicas (por unidade de volume) $p_C = \vec{J}\,\vec{E}$ ou $p_C = \sigma_d E^2$ e as perdas totais de condução do dieléctrico de volume U de valor $P_C = \int_U p_C\, dU$.

Para domínios de temperatura não muito grandes é válida a lei de Van't Hoff da condutividade e portanto para determinados campos eléctricos (ou para uma tensão constante) as perdas variam com a lei exporencial de $\sigma_d$.

De facto, como se deduziu, as perdas relativas à condução caracterizam-se pelo factor de perdas $\operatorname{tg}\delta_C = 1/\omega\varepsilon' R C_o$ ou $\operatorname{tg}\delta_C = \sigma_d/\omega\varepsilon_o\varepsilon'$, quer dizer, as perdas de condução iónica são directa-